

# OPTIMAL ERROR ESTIMATES AND ENERGY CONSERVATION IDENTITIES OF THE ADI-FDTD SCHEME ON STAGGERED GRIDS FOR 3D MAXWELL'S EQUATIONS

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**Abstract.** This paper is concerned with the optimal error estimates and energy conservation properties of the alternating direction implicit finite-difference time-domain (ADI-FDTD) method which is a popular scheme for solving the 3D Maxwell equations. Precisely, for the case with a perfectly electric conducting (PEC) boundary condition we establish the optimal second-order error estimates in both space and time in the discrete  $H^1$ -norm for the ADI-FDTD scheme and prove the approximate divergence preserving property that if the divergence of the initial electric and magnetic fields are zero then the discrete  $L^2$ -norm of the discrete divergence of the ADI-FDTD solution is approximately zero with the second-order accuracy in both space and time. A key ingredient is two new discrete energy norms which are second-order in time perturbations of two new energy conservation laws for the Maxwell equations introduced in this paper. Furthermore, we prove that, in addition to two known discrete energy identities which are second-order in time perturbations of two known energy conservation laws, the ADI-FDTD scheme also satisfies two new discrete energy identities which are second-order in time perturbations of the two new energy conservation laws. This means that the ADI-FDTD scheme is unconditionally stable under the four discrete energy norms. Experimental results are presented which confirm the theoretical results.

**AMS subject classifications.** 65M06, 65M12, 65Z05.

**Key words.** Alternating direction implicit method, finite-difference time-domain method, Maxwell's equations, optimal error estimate, superconvergence, unconditional stability, energy conservation, divergence preserving property.

**1. Introduction.** The alternating direction implicit finite-difference time-domain (ADI-FDTD) method was first proposed in 2000 in [19, 28] for the three-dimensional Maxwell equations with an isotropic medium. This method is based on the staggered grids of Yee's scheme [27] and splitting of Maxwell's equations, consists of only two stages for each time step and is unconditionally stable. The ADI-FDTD scheme is a popular scheme for solving the three-dimensional Maxwell equations and is applicable to a wide range of problems in computational electromagnetics (see [25]). So far, the ADI-FDTD scheme has been intensively studied by many authors (see, e.g. [3, 5, 6, 7, 8, 9, 13, 20, 23, 24, 26, 28, 30]), such as the stability and dispersion analysis using Fourier analysis [6, 8, 20, 23, 28, 30], the unconditional stability and convergence analysis by the energy method [5, 9], its accuracy and efficient algorithms [3, 7, 24, 26] and its applicability with perfectly matched layer absorbing boundary conditions [13]. Meanwhile, motivated by the success of the ADI-FDTD scheme, there is also considerable interest in developing new unconditionally stable schemes for solving the Maxwell equations, based on splitting of the Maxwell equations (see, e.g. [1, 2, 5, 10, 11, 12, 15, 16, 22]), where the stability and convergence of these schemes were studied in the discrete  $L^2$  norm.

However, optimal error estimates and energy conservation and divergence preserving properties of the ADI-FDTD scheme have not yet been studied although these properties are vital for the successful application in practical problems of the ADI-FDTD scheme. The purpose of the present paper is to conduct such a study. In

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particular, for the case of perfectly conducting (PEC) boundary conditions we will establish optimal error estimates in the discrete  $H^1$ -norm for the ADI-FDTD scheme and prove that the ADI-FDTD scheme satisfies four discrete energy identities which are second-order in time perturbations of four energy conservation laws, employing the energy method. We also prove an approximate divergence preserving property with a second-order accuracy in both space and time for the ADI-FDTD scheme. In doing so, a crucial tool is two new discrete energy norms (cf. Theorems 3.2 and 3.3 in Subsection 3.3) which are second-order in time perturbations of two new energy conservation laws for the Maxwell equations (see Theorems 2.1 and 2.2 below) introduced in the present paper.

The remaining part of the paper is organized as follows. In Section 2, we derive two new energy conservation laws for the Maxwell equations with the PEC boundary conditions and introduce two well-known energy conservation laws. In Section 3 we recall the ADI-FDTD scheme, define some new energy norms and prove that the ADI-FDTD scheme satisfies four energy identities which are second-order in time perturbations of the four energy conservation laws for the Maxwell equations introduced in Section 2. This means that the ADI-FDTD scheme is unconditionally stable under the four discrete energy norms. By making use of the two new discrete energy norms we derive optimal second-order error estimates in both space and time under the discrete  $H^1$  norm for the ADI-FDTD scheme (see Theorems 4.2, 4.3 and 4.5) in Section 4 and the approximate divergence preserving property (see Theorem 5.1) in Section 5. Note that it is a superconvergence result that the ADI-FDTD scheme is second-order convergent in space under the discrete  $H^1$  norm. Experimental results are presented in Section 6 which confirm the theoretical results.

**2. Maxwell's equations and their conservation properties.** In this section we introduce four conservation properties of Maxwell's equations, of which two are new. These conservation properties will be used to study the ADI-FDTD scheme in this paper.

**2.1. Maxwell's equations.** Maxwell's equations governing the propagation of electromagnetic waves can be written as a system of partial differential equations (see [25]):

$$(2.1) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = \rho,$$

where  $\mathbf{E}$  is the electric field,  $\mathbf{H}$  is the magnetic field,  $\mathbf{B} = \mu \mathbf{H}$  is the magnetic flux density,  $\mathbf{D} = \varepsilon \mathbf{E}$  is the electric displacement,  $\mathbf{J} = \sigma \mathbf{E}$  is the current density,  $\varepsilon$  is the electric permittivity,  $\mu$  is the magnetic permeability,  $\sigma$  is the electric conductivity and  $\rho$  is the volume density of electric charge.

In this paper we consider Maxwell's equations in a lossless and isotropic medium without a source field. In this case,  $\mu$  and  $\varepsilon$  are constant and  $\rho = \sigma = 0$ , so Maxwell's equations become

$$(2.2) \quad \frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right), \quad \frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right),$$

$$(2.3) \quad \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right), \quad \frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right),$$

$$(2.4) \quad \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right), \quad \frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right),$$

where  $\mathbf{E} = (E_x, E_y, E_z)$  and  $\mathbf{H} = (H_x, H_y, H_z)$  denote the electric and magnetic fields, respectively. For simplicity we consider the perfectly electric conducting (PEC) condition on the boundary  $\partial\Omega$  of the cubic domain  $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ :

$$(2.5) \quad \vec{n} \times \mathbf{E} = 0, \quad \vec{n} \cdot \mathbf{H} = 0, \quad \text{on } (0, T] \times \partial\Omega,$$

where  $T > 0$  and  $\vec{n}$  is the outward normal vector of  $\partial\Omega$ . We also need the initial conditions:

$$(2.6) \quad \mathbf{E}(0, x, y, z) = \mathbf{E}_0(x, y, z) \quad \text{and} \quad \mathbf{H}(0, x, y, z) = \mathbf{H}_0(x, y, z).$$

It is well known that, for suitably smooth data, the problem (2.2)-(2.6) has a unique solution for all time (see [17]).

**2.2. Energy conservation laws in a lossless medium.** When the medium is lossless and isotropic, we have the following two new laws of energy conservation.

**THEOREM 2.1.** (*Energy Conservation I*) *Let  $\mathbf{E}$  and  $\mathbf{H}$  be the solution of the Maxwell equations (2.2) – (2.4) in a lossless and isotropic medium satisfying the PEC boundary conditions (2.5). Then for  $w = x, y, z$ ,*

$$(2.7) \quad \int_{\Omega} \left( \varepsilon \left| \frac{\partial \mathbf{E}}{\partial w} \right|^2 + \mu \left| \frac{\partial \mathbf{H}}{\partial w} \right|^2 \right) dx dy dz \equiv \text{Constant}.$$

*Proof.* We only prove (2.7) for the case with  $w = x$ . The cases with  $w = y, z$  can be shown similarly.

Differentiating the first two equations in (2.1) with respect to  $x$ , we have

$$(2.8) \quad \nabla \times \frac{\partial \mathbf{E}}{\partial x} = -\frac{\partial^2(\mu \mathbf{H})}{\partial x \partial t}, \quad \nabla \times \frac{\partial \mathbf{H}}{\partial x} = \frac{\partial^2(\varepsilon \mathbf{E})}{\partial x \partial t}.$$

Taking the scalar product of the above two equations with  $\partial \mathbf{H} / \partial x$  and  $\partial \mathbf{E} / \partial x$ , respectively, integrating the resulting equations over  $\Omega$  and adding them together, we obtain on integrating by parts that

$$(2.9) \quad \int_{\Omega} \left[ \frac{\partial^2(\mu \mathbf{H})}{\partial x \partial t} \cdot \frac{\partial \mathbf{H}}{\partial x} + \frac{\partial^2(\varepsilon \mathbf{E})}{\partial x \partial t} \cdot \frac{\partial \mathbf{E}}{\partial x} \right] dx dy dz = - \int_{\partial\Omega} (\vec{n} \times \frac{\partial \mathbf{E}}{\partial x}) \cdot \frac{\partial \mathbf{H}}{\partial x} ds.$$

Now, from the PEC boundary condition (2.5) we have  $\vec{n} \times \partial \mathbf{E} / \partial x = 0$  for  $y = 0, 1$  or for  $z = 0, 1$ .

Then (2.9) becomes

$$(2.10) \quad \frac{1}{2} \frac{\partial}{\partial t} \left[ \int_{\Omega} \left( \mu \left| \frac{\partial \mathbf{H}}{\partial x} \right|^2 + \varepsilon \left| \frac{\partial \mathbf{E}}{\partial x} \right|^2 \right) dx dy dz \right] = T_0 + T_1,$$

where

$$T_i = \int_0^1 \int_0^1 \left[ \frac{\partial E_y}{\partial x} \frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial x} \frac{\partial H_y}{\partial x} \right]_{x=i} dy dz, \quad i = 0, 1.$$

Using the Maxwell's equations and the PEC boundary condition it can be shown that  $T_0 = T_1 = 0$ . Using the second equation in (2.2) and the first equation in (2.3) we

have

$$\begin{aligned} T_0 &= \int_0^1 \int_0^1 \left[ \left( \frac{\partial H_x}{\partial z} \frac{\partial E_y}{\partial x} - \frac{\partial H_x}{\partial y} \frac{\partial E_z}{\partial x} \right) - \varepsilon \left( \frac{\partial E_y}{\partial t} \frac{\partial E_y}{\partial x} + \frac{\partial E_z}{\partial t} \frac{\partial E_z}{\partial x} \right) \right]_{x=0} dydz \\ &= \int_0^1 \int_0^1 \left[ \frac{\partial H_x}{\partial z} \frac{\partial E_y}{\partial x} - \frac{\partial H_x}{\partial y} \frac{\partial E_z}{\partial x} \right]_{x=0} dydz \end{aligned}$$

since, by the PEC boundary condition  $\vec{n} \times \mathbf{E} = 0$  on  $\partial\Omega$  in (2.5), we have  $\partial E_y / \partial t = \partial E_z / \partial t = 0$  for  $x = 0$  and for all  $t > 0$ ,  $y \in \Omega$ . By integration by parts it then follows that

$$\begin{aligned} T_0 &= \int_0^1 \left[ H_x \frac{\partial E_y}{\partial x} \Big|_{x=0, z=1} - H_x \frac{\partial E_y}{\partial x} \Big|_{x=0, z=0} \right] dy - \int_0^1 \int_0^1 \left[ H_x \frac{\partial^2 E_y}{\partial z \partial x} \right]_{x=0} dydz \\ &\quad - \int_0^1 \left[ H_x \frac{\partial E_z}{\partial x} \Big|_{x=0, y=1} - H_x \frac{\partial E_z}{\partial x} \Big|_{x=0, y=0} \right] dz - \int_0^1 \int_0^1 \left[ H_x \frac{\partial^2 E_z}{\partial y \partial x} \right]_{x=0} dydz \\ &= 0, \end{aligned}$$

where use has been made of the fact that  $H_x = 0$  for  $x = 0$  and for all  $t > 0$ ,  $y \in \Omega$  in view of the PEC boundary condition  $\vec{n} \cdot \mathbf{H} = 0$  on  $\partial\Omega$  in (2.5).

Similarly, it can be shown that  $T_1 = 0$ . Thus (2.10) implies (2.7) with  $w = x$ . The theorem is thus proved.  $\square$

It is easy to see that the above proof of Theorem 2.1 still works if  $\mathbf{E}$  and  $\mathbf{H}$  are replaced by  $\partial \mathbf{E} / \partial t$  and  $\partial \mathbf{H} / \partial t$ , respectively, so we have the following result.

**THEOREM 2.2.** (*Energy conservation II*) *Let  $\mathbf{E}$  and  $\mathbf{H}$  be the solution of the Maxwell's equations (2.2)-(2.4) in a lossless and isotropic medium satisfying the PEC boundary condition (2.5). Then for  $w = x, y, z$ ,*

$$(2.11) \quad \int \left( \varepsilon \left| \frac{\partial^2 \mathbf{E}}{\partial w \partial t} \right|^2 + \mu \left| \frac{\partial^2 \mathbf{H}}{\partial w \partial t} \right|^2 \right) dx dy dz \equiv \text{Constant}.$$

As far as we know, the two energy conservation laws (2.7) and (2.11) above are new. However, the following two energy conservation laws are already known and were proved in [1]; in particular, the first one is the well-known Poynting theorem and can be found in many classical physical books.

**LEMMA 2.3.** (*Energy conservation III*) *Let  $\mathbf{E}$  and  $\mathbf{H}$  be the solution of the Maxwell's equations (2.2)-(2.4) in a lossless medium satisfying the PEC boundary condition (2.5). Then*

$$(2.12) \quad \int_{\Omega} (\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) dx dy dz \equiv \text{Constant}$$

**LEMMA 2.4.** (*Energy conservation IV*) *Let  $\mathbf{E}$  and  $\mathbf{H}$  be the solution of the Maxwell's equations (2.2)-(2.4) in a lossless medium satisfying the PEC boundary condition (2.5). Then*

$$(2.13) \quad \int_{\Omega} \left( \varepsilon \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2 + \mu \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2 \right) dx dy dz \equiv \text{Constant}$$

We now have four conservation laws (2.7), (2.11), (2.12) and (2.13) for the propagation of electromagnetic waves in a lossless and isotropic medium. We will prove in the next section that the ADI-FDTD scheme keeps these properties with a second-order in time perturbation term.

**3. Energy identities of the ADI-FDTD scheme in  $H^1$ .** Before establishing the energy identities for the ADI-FDTD scheme, we first recall the ADI-FDTD scheme (see [28]) and define some discrete energy norms.

**3.1. The ADI-FDTD scheme.** The ADI-FDTD scheme makes use of Yee's spatial staggered grids (see [27]), which are denoted by  $\Omega^h$  and defined as:

$$\begin{aligned}\Omega^h =: \{ & (x_\alpha, y_\beta, z_\gamma) \mid x_\alpha = \alpha\Delta x, y_\beta = \beta\Delta y, z_\gamma = \gamma\Delta z, \alpha = i, i + 1/2, \\ & i = 0, 1, \dots, I - 1; \beta = j, j + 1/2, j = 0, \dots, J - 1; \gamma = k, k + 1/2, \\ & k = 0, \dots, K - 1; x_0 = y_0 = z_0 = 0, x_I = y_J = z_K = 1\},\end{aligned}$$

where  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  are the mesh sizes along the  $x$ ,  $y$  and  $z$  directions, respectively. In the ADI-FDTD scheme, the field components are defined on different subsets of  $\Omega^h$  (see the subscripts in the equations of this scheme below).

For a positive integer  $N$  let  $\Delta t = T/N$  be the time step and define  $t^n =: n\Delta t$  with  $n = 0, 1, \dots, N$ . Denote by  $u_{\alpha,\beta,\gamma}^m$  the grid function defined on the point  $(m\Delta t, \alpha\Delta x, \beta\Delta y, \gamma\Delta z)$ . Then define

$$\begin{aligned}\delta_x u_{\alpha,\beta,\gamma}^m &= (u_{\alpha+\frac{1}{2},\beta,\gamma}^m - u_{\alpha-\frac{1}{2},\beta,\gamma}^m)/\Delta x, \quad \delta_t u_{\alpha,\beta,\gamma}^m = (u_{\alpha,\beta,\gamma}^{m+\frac{1}{2}} - u_{\alpha,\beta,\gamma}^{m-\frac{1}{2}})/\Delta t, \\ \bar{\delta}_t u_{\alpha,\beta,\gamma}^m &= (u_{\alpha,\beta,\gamma}^{m+\frac{1}{2}} + u_{\alpha,\beta,\gamma}^{m-\frac{1}{2}})/2.\end{aligned}$$

$\delta_y u_{\alpha,\beta,\gamma}^m$  and  $\delta_z u_{\alpha,\beta,\gamma}^m$  can be defined similarly. Denote by  $E_{w\alpha,\beta,\gamma}^m$  and  $H_{w\alpha,\beta,\gamma}^m$  the approximations of the electric field  $E_w(t^m, x_\alpha, y_\beta, z_\gamma)$  and the magnetic field  $H_w(t^m, x_\alpha, y_\beta, z_\gamma)$ ,  $w = x, y, z$ . Then the ADI-FDTD scheme (see [28]) is defined in the following two stages.

Stage 1:

$$(3.1) \quad \frac{E_x^{\bar{n}} - E_x^n}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_y H_z^{\bar{n}} - \delta_z H_y^n \right) \Big|_{\bar{i},j,k}, \quad \frac{E_y^{\bar{n}} - E_y^n}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_z H_x^{\bar{n}} - \delta_x H_z^n \right) \Big|_{i,\bar{j},k},$$

$$(3.2) \quad \frac{E_z^{\bar{n}} - E_z^n}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_x H_y^{\bar{n}} - \delta_y H_x^n \right) \Big|_{i,j,\bar{k}}, \quad \frac{H_x^{\bar{n}} - H_x^n}{\Delta t} = \frac{1}{2\mu} \left( \delta_z E_y^{\bar{n}} - \delta_y E_z^n \right) \Big|_{i,\bar{j},\bar{k}},$$

$$(3.3) \quad \frac{H_y^{\bar{n}} - H_y^n}{\Delta t} = \frac{1}{2\mu} \left( \delta_x E_z^{\bar{n}} - \delta_z E_x^n \right) \Big|_{\bar{i},j,\bar{k}}, \quad \frac{H_z^{\bar{n}} - H_z^n}{\Delta t} = \frac{1}{2\mu} \left( \delta_y E_x^{\bar{n}} - \delta_x E_y^n \right) \Big|_{\bar{i},\bar{j},k},$$

whereafter  $\bar{l} := l + 1/2$  for  $l = n, i, j, k$  and  $F|_{\alpha,\beta,\gamma}$  means each term of the expression or the equation  $F$  has the subscripts  $\alpha, \beta, \gamma$ .

Stage 2:

$$(3.4) \quad \frac{E_x^{n+1} - E_x^{\bar{n}}}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_y H_z^{\bar{n}} - \delta_z H_y^{n+1} \right) \Big|_{\bar{i},j,k}, \quad \frac{E_y^{n+1} - E_y^{\bar{n}}}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_z H_x^{\bar{n}} - \delta_x H_z^{n+1} \right) \Big|_{i,\bar{j},k},$$

$$(3.5) \quad \frac{E_z^{n+1} - E_z^{\bar{n}}}{\Delta t} = \frac{1}{2\varepsilon} \left( \delta_x H_y^{\bar{n}} - \delta_y H_x^{n+1} \right) \Big|_{i,j,\bar{k}}, \quad \frac{H_x^{n+1} - H_x^{\bar{n}}}{\Delta t} = \frac{1}{2\mu} \left( \delta_z E_y^{\bar{n}} - \delta_y E_z^{n+1} \right) \Big|_{i,\bar{j},\bar{k}},$$

$$(3.6) \quad \frac{H_y^{n+1} - H_y^{\bar{n}}}{\Delta t} = \frac{1}{2\mu} \left( \delta_x E_z^{\bar{n}} - \delta_z E_x^{n+1} \right) \Big|_{\bar{i},j,\bar{k}}, \quad \frac{H_z^{n+1} - H_z^{\bar{n}}}{\Delta t} = \frac{1}{2\mu} \left( \delta_y E_x^{\bar{n}} - \delta_x E_y^{n+1} \right) \Big|_{\bar{i},\bar{j},k}.$$

The PEC boundary condition (2.5) can be discretized as:

$$(3.7) \quad \begin{aligned}E_{x_{i+\frac{1}{2},0,k}}^m &= E_{x_{i+\frac{1}{2},J,k}}^m = E_{x_{i+\frac{1}{2},j,0}}^m = E_{x_{i+\frac{1}{2},j,K}}^m = 0, \\ E_{y_{0,j+\frac{1}{2},k}}^m &= E_{y_{I,j+\frac{1}{2},k}}^m = E_{y_{i,j+\frac{1}{2},0}}^m = E_{y_{i,j+\frac{1}{2},K}}^m = 0, \\ E_{z_{0,j,k+\frac{1}{2}}}^m &= E_{z_{I,j,k+\frac{1}{2}}}^m = E_{z_{i,0,k+\frac{1}{2}}}^m = E_{z_{i,J,k+\frac{1}{2}}}^m = 0,\end{aligned}$$

where  $m = n$  or  $n + 1/2$  denotes the time levels and  $i, j, k$  are integers in their valid ranges. We also need the discrete initial conditions which are obtained by imposing the initial condition (2.6) at  $t = 0$ :

$$\mathbf{E}_{\alpha,\beta,\gamma}^0 = \mathbf{E}_0(\alpha\Delta x, \beta\Delta y, \gamma\Delta z), \quad \mathbf{H}_{\alpha,\beta,\gamma}^0 = \mathbf{H}_0(\alpha\Delta x, \beta\Delta y, \gamma\Delta z).$$

**3.2. Discrete energy norms and notations.** We will need some discrete energy norms and notations. For  $\mathbf{U} = (U_{x\bar{i},j,k}, U_{y\bar{i},\bar{j},k}, U_{z\bar{i},j,\bar{k}})$ ,  $\mathbf{V} = (V_{x\bar{i},\bar{j},\bar{k}}, V_{y\bar{i},j,\bar{k}}, V_{z\bar{i},\bar{j},k})$ , define the discrete norms  $\|\cdot\|_E$  and  $\|\cdot\|_H$  as:

$$\begin{aligned} \|\mathbf{U}\|_E^2 &= \left[ \sum_{i=0}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} P(\mathbf{U}) U_{x\bar{i},j,k}^2 + \sum_{i=1}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} P(\mathbf{U}) U_{y\bar{i},\bar{j},k}^2 + \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} P(\mathbf{U}) U_{z\bar{i},j,\bar{k}}^2 \right] \Delta v \\ \|\mathbf{V}\|_H^2 &= \left[ \sum_{i=0}^I \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} P(\mathbf{V}) V_{x\bar{i},\bar{j},\bar{k}}^2 + \sum_{i=0}^{I-1} \sum_{j=0}^J \sum_{k=0}^{K-1} P(\mathbf{V}) V_{y\bar{i},j,\bar{k}}^2 + \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=0}^K P(\mathbf{V}) V_{z\bar{i},\bar{j},k}^2 \right] \Delta v \end{aligned}$$

where  $\Delta v = \Delta x \Delta y \Delta z$ ,  $P(\mathbf{E}) = \varepsilon$ , the electric permittivity, and  $P(\mathbf{H}) = \mu$ , the magnetic permeability. For the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$  we also need the following norms near the boundary grids:

$$\begin{aligned} \|\mathbf{E}\|_I^2 &= \frac{1}{\Delta x} \left[ \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \sum_{i'=1, I-1} \varepsilon E_y^2|_{i', \bar{j}, k} + \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \sum_{i'=1, I-1} \varepsilon E_z^2|_{i', j, \bar{k}} \right] \Delta y \Delta z, \\ \|\mathbf{H}\|_I^2 &= \frac{1}{\Delta x} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \mu \left[ H_x^2|_{1, \bar{j}, \bar{k}} + H_x^2|_{I-1, \bar{j}, \bar{k}} \right] \Delta y \Delta z. \end{aligned}$$

The norms  $\|\mathbf{E}\|_J$ ,  $\|\mathbf{E}\|_K$ ,  $\|\mathbf{H}\|_J$  and  $\|\mathbf{H}\|_K$  are similarly defined by changing the indices  $j, k$ , the factor  $1/\Delta x$  and the area element  $\Delta y \Delta z$  into the corresponding ones.

Furthermore, for simplicity in notations, we introduce some difference operators. For a vector-valued grid function  $\mathbf{U} = (U_{x\alpha,\beta,\gamma}, U_{y\alpha,\beta,\gamma}, U_{z\alpha,\beta,\gamma})$  with  $\alpha = i$  or  $i + 1/2$ ,  $\beta = j$  or  $j + 1/2$ ,  $\gamma = k$  or  $k + 1/2$  define

$$\begin{aligned} \delta_w \mathbf{U} &= (\delta_w U_x, \delta_w U_y, \delta_w U_z)|_{\alpha,\beta,\gamma}, \quad w = x, y, z, \\ \delta_1^h \mathbf{U} &= (\delta_y U_z, \delta_z U_x, \delta_x U_y)|_{\alpha,\beta,\gamma}, \quad \delta_2^h \mathbf{U} = (\delta_z U_y, \delta_x U_z, \delta_y U_x)|_{\alpha,\beta,\gamma}. \end{aligned}$$

By composition of operators it can be easily derived that

$$\begin{aligned} \delta_1^h \delta_1^h \mathbf{U} &= (\delta_y \delta_x U_y, \delta_z \delta_y U_z, \delta_x \delta_z U_x)|_{\alpha,\beta,\gamma}, \\ \delta_2^h \delta_2^h \mathbf{U} &= (\delta_z \delta_x U_z, \delta_x \delta_y U_x, \delta_y \delta_z U_y)|_{\alpha,\beta,\gamma}. \end{aligned}$$

**3.3. Four discrete energy identities for the ADI-FDTD scheme.** We now derive four discrete energy identities for the ADI-FDTD scheme, which are second-order in time perturbations of the four energy conservation laws for the Maxwell equations. To this end, we need the following result on summation by parts.

**LEMMA 3.1.** (*Summation by Parts*) Let  $M \geq 1$  be an integer. For any sequence  $\{W_{m+1/2}\}_{m=0}^{M-1}$  let  $\delta$  is a difference operator defined by  $\delta W_m = (W_{m+1/2} - W_{m-1/2})/h$ , where  $h$  is a positive number. Then, for any two sequences  $\{U_{m+1/2}\}_{m=0}^{M-1}$  and  $\{V_m\}_{m=0}^M$  we have

$$(3.8) \quad \sum_{m=0}^{M-1} U_{m+1/2} \delta V_{m+1/2} = U_{M-1/2} V_M - U_{1/2} V_0 - \sum_{m=1}^{M-1} V_m \delta U_m.$$

**THEOREM 3.2.** (*Energy identity I*) Let  $n \geq 0$  and let  $\mathbf{E}^n = (E_{x\bar{i},j,k}^n, E_{y\bar{i},j,k}^n, E_{z\bar{i},j,k}^n)$ ,  $\mathbf{H}^n = (H_{x\bar{i},j,k}^n, H_{y\bar{i},j,k}^n, H_{z\bar{i},j,k}^n)$  be the solution of the ADI-FDTD scheme (3.1) – (3.6) with the boundary condition (3.7). Then we have the following energy identities:

$$\begin{aligned}
& \|\delta_w \mathbf{E}^{n+1}\|_E^2 + \|\delta_w \mathbf{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_w \delta_1^h \mathbf{E}^{n+1}\|_H^2 + \|\delta_w \delta_2^h \mathbf{H}^{n+1}\|_E^2 \right) \\
& + \|\mathbf{E}^{n+1}\|_{L(w)}^2 + \|\mathbf{H}^{n+1}\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \mathbf{H}^{n+1}\|_{L(w)}^2 + \|\delta_1^h \mathbf{E}^{n+1}\|_{L(w)}^2 \right) \\
& = \|\delta_w \mathbf{E}^n\|_E^2 + \|\delta_w \mathbf{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_w \delta_1^h \mathbf{E}^n\|_H^2 + \|\delta_w \delta_2^h \mathbf{H}^n\|_E^2 \right) \\
(3.9) \quad & + \|\mathbf{E}^n\|_{L(w)}^2 + \|\mathbf{H}^n\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \mathbf{H}^n\|_{L(w)}^2 + \|\delta_1^h \mathbf{E}^n\|_{L(w)}^2 \right),
\end{aligned}$$

where  $w = x, y, z$ ,  $L(x) = I$ ,  $L(y) = J$ ,  $L(z) = K$ .

*Proof.* We only prove (3.9) for the case with  $w = x$ . The other cases with  $w = y, z$  can be proved similarly.

Applying the difference operator  $\delta_x$  to each equation in Stages 1 of the ADI-FDTD scheme, rearranging the terms according to the time levels and squaring both sides of the six equations thus obtained, we have

$$\begin{aligned}
(3.10) \quad & \varepsilon(\delta_x E_x^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_y H_z^{n+\frac{1}{2}})^2 - \Delta t \delta_x E_x^{n+\frac{1}{2}} \delta_x \delta_y H_z^{n+\frac{1}{2}} \\
& = \varepsilon(\delta_x E_x^n)^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_z H_y^n)^2 - \Delta t \delta_x E_x^n \delta_x \delta_z H_y^n \Big|_{i,j,k},
\end{aligned}$$

$$\begin{aligned}
(3.11) \quad & \varepsilon(\delta_x E_y^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_z H_x^{n+\frac{1}{2}})^2 - \Delta t \delta_x E_y^{n+\frac{1}{2}} \delta_x \delta_z H_x^{n+\frac{1}{2}} \\
& = \varepsilon(\delta_x E_y^n)^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_x H_z^n)^2 - \Delta t \delta_x E_y^n \delta_x \delta_x H_z^n \Big|_{i+\frac{1}{2},j+\frac{1}{2},k},
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad & \varepsilon(\delta_x E_z^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_x H_y^{n+\frac{1}{2}})^2 - \Delta t \delta_x E_z^{n+\frac{1}{2}} \delta_x \delta_x H_y^{n+\frac{1}{2}} \\
& = \varepsilon(\delta_x E_z^n)^2 + \frac{(\Delta t)^2}{4\varepsilon} (\delta_x \delta_y H_x^n)^2 - \Delta t \delta_x E_z^n \delta_x \delta_y H_x^n \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
(3.13) \quad & \mu(\delta_x H_x^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_z E_y^{n+\frac{1}{2}})^2 - \Delta t \delta_x H_x^{n+\frac{1}{2}} \delta_x \delta_z E_y^{n+\frac{1}{2}} \\
& = \mu(\delta_x H_x^n)^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_y E_z^n)^2 - \Delta t \delta_x H_x^n \delta_x \delta_y E_z^n \Big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
(3.14) \quad & \mu(\delta_x H_y^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_x E_z^{n+\frac{1}{2}})^2 - \Delta t \delta_x H_y^{n+\frac{1}{2}} \delta_x \delta_x E_z^{n+\frac{1}{2}} \\
& = \mu(\delta_x H_y^n)^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_z E_x^n)^2 - \Delta t \delta_x H_y^n \delta_x \delta_z E_x^n \Big|_{i,j,k+\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
(3.15) \quad & \mu(\delta_x H_z^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_y E_x^{n+\frac{1}{2}})^2 - \Delta t \delta_x H_z^{n+\frac{1}{2}} \delta_x \delta_y E_x^{n+\frac{1}{2}} \\
& = \mu(\delta_x H_z^n)^2 + \frac{(\Delta t)^2}{4\mu} (\delta_x \delta_x E_y^n)^2 - \Delta t \delta_x H_z^n \delta_x \delta_x E_y^n \Big|_{i,j+\frac{1}{2},k}.
\end{aligned}$$

From the definition of  $\delta_x$  and the conditions (3.7) it follows that

$$(3.16) \quad \delta_x E_{x_{i,j',k}}^m = \delta_x E_{x_{i,j,k'}}^m = \delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k'}}^m = \delta_x E_{z_{i+\frac{1}{2},j',k+\frac{1}{2}}}^m = 0$$

where  $m = n, n + 1/2, j' = 0, J, k' = 0, K$ . By these boundary conditions and summation by parts (Lemma 3.1), we can see that the sum of all the mixed-product terms such as  $\delta_x E_u \delta_x \delta_u H_v$  ( $u, v = x, y, z$ ) on the left- and right-hand sides of (3.10)-(3.15) will vanish, respectively, due to cancelation. For example, consider the sum of the mixed-product terms on the left hand sides of (3.15), (3.13) and (3.14) over  $i, j$  and  $k$  in their valid range. We have

$$\begin{aligned} & \sum_{i=1}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \delta_x H_z^{\bar{n}} \delta_x \delta_y E_x^{\bar{n}} \Big|_{i,j+\frac{1}{2},k} = - \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \delta_x E_x^{\bar{n}} \delta_y \delta_x H_z^{\bar{n}} \Big|_{i,j,k}, \\ & \sum_{i=1}^{I-2} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \delta_x H_x^{\bar{n}} \delta_x \delta_z E_y^{\bar{n}} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = - \sum_{i=1}^{I-2} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \delta_x E_y^{\bar{n}} \delta_z \delta_x H_x^{\bar{n}} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}, \\ & \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \delta_x H_y^{\bar{n}} \delta_x \delta_x E_z^{\bar{n}} \Big|_{i,j,k+\frac{1}{2}} = - \sum_{i=1}^{I-2} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \delta_x E_z^{\bar{n}} \delta_x \delta_x H_y^{\bar{n}} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}} \\ & + \frac{1}{\Delta x} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left\{ \delta_x E_z^{\bar{n}} \Big|_{I-\frac{1}{2},j,k+\frac{1}{2}} \delta_x H_y^{\bar{n}} \Big|_{I-1,j,k+\frac{1}{2}} - \delta_x E_z^{\bar{n}} \Big|_{\frac{1}{2},j,k+\frac{1}{2}} \delta_x H_y^{\bar{n}} \Big|_{1,j,k+\frac{1}{2},k+\frac{1}{2}} \right\}. \end{aligned}$$

The sum of the above three terms can cancel the sums of the three mixed-product terms on the left-hand side of (3.10), (3.11) and (3.12), respectively. Thus, sum up each of the six equations above over the valid ranges of their subscripts  $i, j, k$  and add the updated six equations together to deduce that

$$\begin{aligned} & \|\delta_x \mathbf{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_x \mathbf{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_2^h \mathbf{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_x \delta_1^h \mathbf{H}^{n+\frac{1}{2}}\|_E^2) \\ & = \|\delta_x \mathbf{E}^n\|_E^2 + \|\delta_x \mathbf{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathbf{E}^n\|_H^2 + \|\delta_x \delta_2^h \mathbf{H}^n\|_E^2) \\ & \quad - \frac{\Delta t}{\Delta x} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ E_z^{n+\frac{1}{2}} \delta_x H_y^{n+\frac{1}{2}} \Big|_{I-1,j,k+\frac{1}{2}} + E_z^{n+\frac{1}{2}} \delta_x H_y^{n+\frac{1}{2}} \Big|_{1,j,k+\frac{1}{2}} \right] \Delta y \Delta z \\ (3.17) \quad & + \frac{\Delta t}{\Delta x} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left[ E_y^n \delta_x H_z^n \Big|_{I-1,j+\frac{1}{2},k} + E_y^n \delta_x H_z^n \Big|_{1,j+\frac{1}{2},k} \right] \Delta y \Delta z, \end{aligned}$$

where we have used the boundary conditions (3.7) applied by the operator  $\delta_x$  to get the last two terms on the right-hand side of the above equation.

Similarly, from the equations in Stage 2 applied by the operator  $\delta_x$  we can derive that

$$\begin{aligned} & \|\delta_x \mathbf{E}^{n+1}\|_E^2 + \|\delta_x \mathbf{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathbf{E}^{n+1}\|_H^2 + \|\delta_x \delta_2^h \mathbf{H}^{n+1}\|_E^2) \\ & = \|\delta_x \mathbf{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_x \mathbf{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_2^h \mathbf{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_x \delta_1^h \mathbf{H}^{n+\frac{1}{2}}\|_E^2) \\ & \quad - \frac{\Delta t}{\Delta x} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left( E_z^{n+\frac{1}{2}} \delta_x H_y^{n+\frac{1}{2}} \Big|_{I-1,j,k+\frac{1}{2}} + E_z^{n+\frac{1}{2}} \delta_x H_y^{n+\frac{1}{2}} \Big|_{1,j,k+\frac{1}{2}} \right) \Delta y \Delta z \\ (3.18) \quad & + \frac{\Delta t}{\Delta x} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left( E_y^{n+1} \delta_x H_z^{n+1} \Big|_{I-1,j+\frac{1}{2},k} + E_y^{n+1} \delta_x H_z^{n+1} \Big|_{1,j+\frac{1}{2},k} \right) \Delta y \Delta z. \end{aligned}$$



We now consider the mixed-product terms on the right-hand side of (3.17) and (3.18). Taking  $i = 1$  and  $I - 1$  in the first equation in (3.2) and (3.5) we have

$$\begin{aligned} E_{z_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}} - \frac{\Delta t}{2\varepsilon} \delta_x H_{y_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}} &= E_{z_{i',j,k+\frac{1}{2}}}^n - \frac{\Delta t}{2\varepsilon} \delta_y H_{x_{i',j,k+\frac{1}{2}}}^n, \\ E_{z_{i',j,k+\frac{1}{2}}}^{n+1} + \frac{\Delta t}{2\varepsilon} \delta_y H_{x_{i',j,k+\frac{1}{2}}}^{n+1} &= E_{z_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}} + \frac{\Delta t}{2\varepsilon} \delta_x H_{y_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}}, \end{aligned}$$

where  $i' = 1, I - 1$ . Multiplying both sides of the above two equations by  $\sqrt{\varepsilon}$ , squaring them and adding the resulting equations give

$$\begin{aligned} 2\Delta t E_{z_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}} \delta_x H_{y_{i',j,k+\frac{1}{2}}}^{n+\frac{1}{2}} &= \varepsilon(E_z^{n+1})^2 - \varepsilon(E_z^n)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left[ \mu(\delta_y H_x^{n+1})^2 - \mu(\delta_y H_x^n)^2 \right] \\ (3.19) \quad &+ \Delta t \left( E_z^{n+1} \delta_y H_x^{n+1} + E_z^n \delta_y H_x^n \right) \Big|_{i',j,k+\frac{1}{2}}, \quad i' = 1, I - 1. \end{aligned}$$

Similarly, it follows from the second equation in (3.1) and (3.4) with  $i = 1, I - 1$  that

$$\begin{aligned} \Delta t \left( E_y^n \delta_x H_z^n + E_y^{n+1} \delta_x H_z^{n+1} \right) &= \varepsilon(E_y^n)^2 - \varepsilon(E_y^{n+1})^2 + 2\Delta t E_y^{n+\frac{1}{2}} \delta_z H_x^{n+\frac{1}{2}} \\ (3.20) \quad &+ \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \mu(\delta_x H_z^n)^2 - \mu(\delta_x H_z^{n+1})^2 \right) \Big|_{i',j+\frac{1}{2},k}, \quad i' = 1, I - 1. \end{aligned}$$

Combining (3.17), (3.18), (3.19) and (3.20) gives

$$\begin{aligned} &\|\delta_x \mathbf{E}^{n+1}\|_E^2 + \|\delta_x \mathbf{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathbf{E}^{n+1}\|_H^2 + \|\delta_x \delta_2^h \mathbf{H}^{n+1}\|_E^2) \\ &+ \frac{1}{\Delta x} \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ \varepsilon(E_z^{n+1})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_y H_x^{n+1})^2 \right]_{i',j,k+\frac{1}{2}} \Delta y \Delta z \\ &+ \frac{1}{\Delta x} \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left[ \varepsilon(E_y^{n+1})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x H_z^{n+1})^2 \right]_{i',j+\frac{1}{2},k} \Delta y \Delta z \\ &= \|\delta_x \mathbf{E}^n\|_E^2 + \|\delta_x \mathbf{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathbf{E}^n\|_H^2 + \|\delta_x \delta_2^h \mathbf{H}^n\|_E^2) \\ &+ \frac{1}{\Delta x} \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ \varepsilon(E_z^n)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_y H_x^n)^2 \right]_{i',j,k+\frac{1}{2}} \Delta y \Delta z \\ (3.21) \quad &+ \frac{1}{\Delta x} \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left[ \varepsilon(E_y^n)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x H_z^n)^2 \right]_{i',j+\frac{1}{2},k} \Delta y \Delta z + R, \end{aligned}$$

where the term  $R$  is given by

$$\begin{aligned} R &= -\frac{\Delta t}{\Delta x} \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ E_z^{n+1} \delta_y H_x^{n+1} + E_z^n \delta_y H_x^n \right]_{i',j,k+\frac{1}{2}} \Delta y \Delta z \\ &+ \frac{2\Delta t}{\Delta x} \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} E_y^{n+\frac{1}{2}} \delta_z H_x^{n+\frac{1}{2}} \Big|_{i',j+\frac{1}{2},k} \Delta y \Delta z. \end{aligned}$$

Arguing similarly as in deriving (3.19) we deduce from the second equation in (3.2) and (3.5) with  $i = i'$  that

$$(3.22) \quad R = \frac{1}{\Delta x} \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left[ \mu(H_x^n)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \varepsilon (\delta_y E_z^n)^2 - \left\{ \mu(H_x^{n+1})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \varepsilon (\delta_y E_z^{n+1})^2 \right\} \right]_{i', j+\frac{1}{2}, k+\frac{1}{2}} \Delta y \Delta z.$$

Substituting (3.22) into (3.21) gives the identity (3.9) with  $w = x$ , noting the definition of the norms  $\|\mathbf{E}\|_I$  and  $\|\mathbf{H}\|_I$ . The proof is thus complete.  $\square$

Note that the proof of Theorem 3.2 does not depend on the time levels. Thus, if we apply the operators  $\delta_t$  and  $\delta_w \delta_t$  with  $w = x, y, z$  to the equations in the ADI-FDTD scheme and repeat the above argument, then we can obtain the following result.

**THEOREM 3.3. (Energy identities II)** *Let  $n \geq 1$  and let  $\mathbf{E}^n, \mathbf{H}^n$  be the solution of the ADI-FDTD scheme. Then*

$$\begin{aligned} & \|\delta_w \delta_t \mathbf{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_w \delta_t \mathbf{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_w \delta_1^h \delta_t \mathbf{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_w \delta_2^h \delta_t \mathbf{H}^{n+\frac{1}{2}}\|_E^2 \right) \\ & + \|\delta_t \mathbf{E}^{n+\frac{1}{2}}\|_{L(w)}^2 + \|\delta_t \mathbf{H}^{n+\frac{1}{2}}\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \delta_t \mathbf{H}^{n+\frac{1}{2}}\|_{L(w)}^2 + \|\delta_1^h \delta_t \mathbf{E}^{n+\frac{1}{2}}\|_{L(w)}^2 \right) \\ & = \|\delta_w \delta_t \mathbf{E}^{n-\frac{1}{2}}\|_E^2 + \|\delta_w \delta_t \mathbf{H}^{n-\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_w \delta_1^h \delta_t \mathbf{E}^{n-\frac{1}{2}}\|_H^2 + \|\delta_w \delta_2^h \delta_t \mathbf{H}^{n-\frac{1}{2}}\|_E^2 \right) \\ & + \|\delta_t \mathbf{E}^{n-\frac{1}{2}}\|_{L(w)}^2 + \|\delta_t \mathbf{H}^{n-\frac{1}{2}}\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \delta_t \mathbf{H}^{n-\frac{1}{2}}\|_{L(w)}^2 + \|\delta_1^h \delta_t \mathbf{E}^{n-\frac{1}{2}}\|_{L(w)}^2 \right), \end{aligned}$$

where  $w = x, y, z$ ,  $L(x) = I$ ,  $L(y) = J$  and  $L(z) = K$ .

Theorems 3.2 and 3.3 are reduced to the following two results, respectively, when  $\delta_x = I$  (identity operator).

**THEOREM 3.4. (Energy identity III)** *Let  $n \geq 0$  and let  $\mathbf{E}^n, \mathbf{H}^n$  be the solution of the ADI-FDTD scheme. Then*

$$\begin{aligned} & \|\mathbf{E}^{n+1}\|_E^2 + \|\mathbf{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \mathbf{H}^{n+1}\|_E^2 + \|\delta_1^h \mathbf{E}^{n+1}\|_H^2 \right) \\ & = \|\mathbf{E}^n\|_E^2 + \|\mathbf{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \mathbf{H}^n\|_E^2 + \|\delta_1^h \mathbf{E}^n\|_H^2 \right). \end{aligned}$$

**THEOREM 3.5. (Energy identity IV)** *Let  $n \geq 1$  and let  $\mathbf{E}^n, \mathbf{H}^n$  be the solution of the ADI-FDTD scheme. Then the ADI-FDTD scheme satisfies the following identity:*

$$\begin{aligned} & \|\delta_t \mathbf{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_t \mathbf{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \delta_t \mathbf{H}^{n+\frac{1}{2}}\|_E^2 + \|\delta_1^h \delta_t \mathbf{E}^{n+\frac{1}{2}}\|_H^2 \right) \\ & = \|\delta_t \mathbf{E}^{n-\frac{1}{2}}\|_E^2 + \|\delta_t \mathbf{H}^{n-\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \delta_t \mathbf{H}^{n-\frac{1}{2}}\|_E^2 + \|\delta_1^h \delta_t \mathbf{E}^{n-\frac{1}{2}}\|_H^2 \right). \end{aligned}$$

**COROLLARY 3.6.** *The ADI-FDTD scheme (3.1) – (3.6) with the PEC boundary condition (3.7) is unconditionally stable under the new defined discrete energy norms and under the discrete  $H^1$  norm.*

**4. Optimal error estimates for the ADI-FDTD scheme.** In this section, we derive optimal error estimates for the ADI-FDTD scheme under the discrete energy norms. We need the following well-known result called the discrete Gronwall's lemma (see [14]).

LEMMA 4.1. *If  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are three positive sequences with  $\{c_n\}$  being monotone such that  $a_n + b_n \leq c_n + \lambda \sum_{i=0}^{n-1} a_i$  with  $\lambda > 0$  and  $a_0 + b_0 \leq c_0$  then  $a_n + b_n \leq c_n \exp(n\lambda)$  for all  $n \geq 0$ .*

For  $n \geq 0$ ,  $w = x, y, z$  and for  $\alpha = i, \bar{i}$ ,  $\beta = j, \bar{j}$ ,  $\gamma = k, \bar{k}$  let  $\mathcal{E}_{w_{\alpha,\beta,\gamma}}^n = e_w(t^n, x_\alpha, y_\beta, z_\gamma) - E_{w_{\alpha,\beta,\gamma}}^n$ ,  $\mathcal{H}_{w_{\alpha,\beta,\gamma}}^n = h_w(t^n, x_\alpha, y_\beta, z_\gamma) - H_{w_{\alpha,\beta,\gamma}}^n$ ,  $\mathcal{E}^n = (\mathcal{E}_x^n, \mathcal{E}_y^n, \mathcal{E}_z^n)$ ,  $\mathcal{H}^n = (\mathcal{H}_x^n, \mathcal{H}_y^n, \mathcal{H}_z^n)$ , where  $\mathbf{e} = (e_x, e_y, e_z)$ ,  $\mathbf{h} = (h_x, h_y, h_z)$  is the exact solution to the Maxwell equations (2.2)-(2.4) with the boundary condition (2.5) and the initial condition (2.6) and  $\mathbf{E}^n = (E_{x_{i,j,k}}^n, E_{y_{i,j,k}}^n, E_{z_{i,j,k}}^n)$ ,  $\mathbf{H}^n = (H_{x_{i,j,k}}^n, H_{y_{i,j,k}}^n, H_{z_{i,j,k}}^n)$  is the solution of the ADI-FDTD scheme.

THEOREM 4.2. *Let the exact solution  $\mathbf{e}$ ,  $\mathbf{h}$  satisfy that  $\mathbf{e} \in C((0, T]; C^4(\bar{\Omega})) \cap C^1((0, T]; C^2(\bar{\Omega})) \cap C^2((0, T]; C^1(\bar{\Omega})) \cap C^3((0, T]; C(\bar{\Omega}))$ ,  $\mathbf{h} \in C((0, T]; C^4(\bar{\Omega})) \cap C^1((0, T]; C^2(\bar{\Omega})) \cap C^2((0, T]; C^1(\bar{\Omega})) \cap C^3((0, T]; C(\bar{\Omega}))$ . Then for any  $n \geq 0$  and any fixed  $T > 0$  there is constant  $C$  independent of  $\Delta t, \Delta x, \Delta y, \Delta z$  such that*

$$(4.1) \quad \|\delta_w \mathcal{E}^{n+1}\|_E^2 + \|\delta_w \mathcal{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_w \delta_1^h \mathcal{E}^{n+1}\|_H^2 + \|\delta_w \delta_2^h \mathcal{H}^{n+1}\|_E^2) \\ + \|\mathcal{E}^{n+1}\|_{L(w)}^2 + \|\mathcal{H}^{n+1}\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} [\|\delta_2^h \mathcal{H}^{n+1}\|_{L(w)}^2 + \|\delta_1^h \mathcal{E}^{n+1}\|_{L(w)}^2] \\ \leq C\{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\},$$

where  $w = x, y, z$ ,  $L(x) = I$ ,  $L(y) = J$  and  $L(z) = K$ .

*Proof.* We only prove (4.1) for the case with  $w = x$ . The other cases with  $w = y, z$  can be proved similarly.

In order to derive the error equations for the ADI-FDTD scheme and the  $\delta_x$ -ADI-FDTD scheme (i.e. the ADI-FDTD scheme applied by  $\delta_x$ ), we need two discrete forms of the Maxwell equations corresponding to the two schemes. Denote by  $e_w^{n*}$ ,  $h_w^{n*}$  ( $w = x, y, z$ ) the intermediate variables defined by

$$\begin{aligned} \delta_x e_x^{n*} &= \frac{1}{2} \delta_x (e_x^{n+1} + e_x^n) + \frac{\Delta t}{4\varepsilon} \delta_x \delta_z (h_y^{n+1} - h_y^n)|_{i,j,k}, \\ \delta_x e_y^{n*} &= \frac{1}{2} \delta_x (e_y^{n+1} + e_y^n) + \frac{\Delta t}{4\varepsilon} \delta_x \delta_x (h_z^{n+1} - h_z^n)|_{i+\frac{1}{2}, j+\frac{1}{2}, k}, \\ \delta_x e_z^{n*} &= \frac{1}{2} \delta_x (e_z^{n+1} + e_z^n) + \frac{\Delta t}{4\varepsilon} \delta_x \delta_y (h_x^{n+1} - h_x^n)|_{i+\frac{1}{2}, j, k+\frac{1}{2}}, \\ \delta_x h_x^{n*} &= \frac{1}{2} \delta_x (h_x^{n+1} + h_x^n) + \frac{\Delta t}{4\mu} \delta_x \delta_y (e_z^{n+1} - e_z^n)|_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}, \\ \delta_x h_y^{n*} &= \frac{1}{2} \delta_x (h_y^{n+1} + h_y^n) + \frac{\Delta t}{4\mu} \delta_x \delta_z (e_x^{n+1} - e_x^n)|_{i, j, k+\frac{1}{2}}, \\ \delta_x h_z^{n*} &= \frac{1}{2} \delta_x (h_z^{n+1} + h_z^n) + \frac{\Delta t}{4\mu} \delta_x \delta_x (e_y^{n+1} - e_y^n)|_{i, j+\frac{1}{2}, k}. \end{aligned}$$

Then by a direct calculation we derive that

Stage 1 of the discrete form of the Maxwell equations:

$$(4.2) \quad \frac{\delta_x e_x^{n*} - \delta_x e_x^n}{\Delta t/2} = \frac{1}{\varepsilon} (\delta_y \delta_x h_z^{n*} - \delta_z \delta_x h_y^n) + \delta_x \beta_x^{n+\frac{1}{2}} \Big|_{i,j,k},$$

$$(4.3) \quad \frac{\delta_x e_y^{n*} - \delta_x e_y^n}{\Delta t/2} = \frac{1}{\varepsilon} \left( \delta_z \delta_x h_x^{n*} - \delta_x \delta_x h_z^n \right) + \delta_x \beta_y^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j+\frac{1}{2}, k},$$

$$(4.4) \quad \frac{\delta_x e_z^{n*} - \delta_x e_z^n}{\Delta t/2} = \frac{1}{\varepsilon} \left( \delta_x \delta_x h_y^{n*} - \delta_y \delta_x h_x^n \right) + \delta_x \beta_z^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j, k+\frac{1}{2}},$$

$$(4.5) \quad \frac{\delta_x h_x^{n*} - \delta_x h_x^n}{\Delta t/2} = \frac{1}{\mu} \left( \delta_z \delta_x e_y^{n*} - \delta_y \delta_x e_z^n \right) + \delta_x \xi_x^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}},$$

$$(4.6) \quad \frac{\delta_x h_y^{n*} - \delta_x h_y^n}{\Delta t/2} = \frac{1}{\mu} \left( \delta_x \delta_x e_z^{n*} - \delta_z \delta_x e_x^n \right) + \delta_x \xi_y^{n+\frac{1}{2}} \Big|_{i, j, k+\frac{1}{2}},$$

$$(4.7) \quad \frac{\delta_x h_z^{n*} - \delta_x h_z^n}{\Delta t/2} = \frac{1}{\mu} \left( \delta_y \delta_x e_x^{n*} - \delta_x \delta_x e_y^n \right) + \delta_x \xi_z^{n+\frac{1}{2}} \Big|_{i, j+\frac{1}{2}, k}.$$

Stage 2 of the discrete form of the Maxwell equations:

$$(4.8) \quad \frac{\delta_x e_x^{n+1} - \delta_x e_x^{n*}}{\Delta t/2} = \frac{1}{\varepsilon} \left( \delta_y \delta_x h_z^{n*} - \delta_z \delta_x h_y^{n+1} \right) + \delta_x \beta_x^{n+\frac{1}{2}} \Big|_{i, j, k},$$

$$(4.9) \quad \frac{\delta_x e_y^{n+1} - \delta_x e_y^{n*}}{\Delta t/2} = \frac{1}{\varepsilon} \left( \delta_z \delta_x h_x^{n*} - \delta_x \delta_x h_z^{n+1} \right) + \delta_x \beta_y^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j+\frac{1}{2}, k},$$

$$(4.10) \quad \frac{\delta_x e_z^{n+1} - \delta_x e_z^{n*}}{\Delta t/2} = \frac{1}{\varepsilon} \left( \delta_x \delta_x h_y^{n*} - \delta_y \delta_x h_x^{n+1} \right) + \delta_x \beta_z^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j, k+\frac{1}{2}},$$

$$(4.11) \quad \frac{\delta_x h_x^{n+1} - \delta_x h_x^{n*}}{\Delta t/2} = \frac{1}{\mu} \left( \delta_z \delta_x e_y^{n*} - \delta_y \delta_x e_z^{n+1} \right) + \delta_x \xi_x^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}},$$

$$(4.12) \quad \frac{\delta_x h_y^{n+1} - \delta_x h_y^{n*}}{\Delta t/2} = \frac{1}{\mu} \left( \delta_x \delta_x e_z^{n*} - \delta_z \delta_x e_x^{n+1} \right) + \delta_x \xi_y^{n+\frac{1}{2}} \Big|_{i, j, k+\frac{1}{2}},$$

$$(4.13) \quad \frac{\delta_x h_z^{n+1} - \delta_x h_z^{n*}}{\Delta t/2} = \frac{1}{\mu} \left( \delta_y \delta_x e_x^{n*} - \delta_x \delta_x e_y^{n+1} \right) + \delta_x \xi_z^{n+\frac{1}{2}} \Big|_{i, j+\frac{1}{2}, k}.$$

Here, in Stage 1 and Stage 2 of the discrete form of the Maxwell equations  $\delta_x \beta_w^{n+\frac{1}{2}}$  and  $\delta_x \xi_w^{n+\frac{1}{2}}$  ( $w = x, y, z$ ) are the truncation error terms of the equivalent form of the  $\delta_x$ -ADI-FDTD scheme (see below). For example,

$$\begin{aligned} \delta_x \beta_x^{n+\frac{1}{2}} &= \delta_x \delta_t e_x^{n+\frac{1}{2}} - \frac{1}{\varepsilon} \delta_x \bar{\delta}_t \left[ \delta_y h_z^{n+\frac{1}{2}} - \delta_z h_y^{n+\frac{1}{2}} \right] - \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_y \delta_x \delta_t e_y^{n+\frac{1}{2}} \Big|_{i, j, k}, \\ \delta_x \xi_x^{n+\frac{1}{2}} &= \delta_x \delta_t h_x^{n+\frac{1}{2}} - \frac{1}{\mu} \delta_x \bar{\delta}_t \left[ \delta_z e_y^{n+\frac{1}{2}} - \delta_y e_z^{n+\frac{1}{2}} \right] - \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_z \delta_x \delta_t h_z^{n+\frac{1}{2}} \Big|_{i, j, k}. \end{aligned}$$

The other terms can be obtained similarly. Then the equivalent form of the  $\delta_x$ -ADI-FDTD scheme can be easily derived by combining all the equations in the  $\delta_x$ -ADI-FDTD scheme and canceling the field values at the intermediate time levels  $t^{n+\frac{1}{2}}$  as follows:

$$\begin{aligned} \delta_x \delta_t E_x^{n+\frac{1}{2}} &= \frac{1}{\varepsilon} \delta_x \bar{\delta}_t \left[ \delta_y H_z^{n+\frac{1}{2}} - \delta_z H_y^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_y \delta_x \delta_t E_y^{n+\frac{1}{2}} \Big|_{i, j, k}, \\ \delta_x \delta_t E_y^{n+\frac{1}{2}} &= \frac{1}{\varepsilon} \delta_x \bar{\delta}_t \left[ \delta_z H_x^{n+\frac{1}{2}} - \delta_x H_z^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_z \delta_y \delta_t E_z^{n+\frac{1}{2}} \Big|_{i, j, k}, \\ \delta_x \delta_t E_z^{n+\frac{1}{2}} &= \frac{1}{\varepsilon} \delta_x \bar{\delta}_t \left[ \delta_x H_y^{n+\frac{1}{2}} - \delta_y H_x^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_x \delta_z \delta_t E_x^{n+\frac{1}{2}} \Big|_{i, j, k}; \\ \delta_x \delta_t H_x^{n+\frac{1}{2}} &= \frac{1}{\mu} \delta_x \bar{\delta}_t \left[ \delta_z E_y^{n+\frac{1}{2}} - \delta_y E_z^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_z \delta_x \delta_t H_z^{n+\frac{1}{2}} \Big|_{i, j, k}, \end{aligned}$$

$$\begin{aligned}\delta_x \delta_t H_y^{n+\frac{1}{2}} &= \frac{1}{\mu} \delta_x \bar{\delta}_t \left[ \delta_x E_z^{n+\frac{1}{2}} - \delta_z E_x^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_x \delta_y \delta_t H_x^{n+\frac{1}{2}} \Big|_{i,j,\bar{k}}, \\ \delta_x \delta_t H_z^{n+\frac{1}{2}} &= \frac{1}{\mu} \delta_x \bar{\delta}_t \left[ \delta_y E_x^{n+\frac{1}{2}} - \delta_x E_y^{n+\frac{1}{2}} \right] + \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_x \delta_y \delta_z \delta_t H_y^{n+\frac{1}{2}} \Big|_{i,\bar{j},k}.\end{aligned}$$

Removing the operator  $\delta_x$  from all the terms in the above equations leads to the equivalent form of the ADI-FDTD scheme which is omitted here for shortness.

From the above equivalent scheme and the Taylor expansion it can be seen that the truncation terms in (4.2)-(4.13) are bounded if the exact solution of the Maxwell equations is suitably smooth, that is,

$$(4.14) \quad |\delta_x \beta_{x_{i,j,k}}^{\bar{n}}| + |\delta_x \beta_{y_{i,\bar{j},k}}^{\bar{n}}| + |\delta_x \beta_{z_{i,\bar{j},\bar{k}}}^{\bar{n}}| \leq C_{eh} \{(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\},$$

$$(4.15) \quad |\delta_x \xi_{x_{i,\bar{j},\bar{k}}}^{\bar{n}}| + |\delta_x \xi_{y_{i,j,\bar{k}}}^{\bar{n}}| + |\delta_x \xi_{z_{i,\bar{j},k}}^{\bar{n}}| \leq C_{eh} \{(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\},$$

where  $\bar{l} = l + \frac{1}{2}$  for  $l = i, j, k, n$  and  $C_{eh}$  is a positive constant depending on  $\varepsilon, \mu$  and the upper bounds of the derivatives of  $\mathbf{e}, \mathbf{h}$ .

For  $n \geq 0$ ,  $w = x, y, z$  and for  $\alpha = i, i + 1/2$ ,  $\beta = j, j + 1/2$ ,  $\gamma = k, k + 1/2$  let

$$\begin{aligned}\delta_x \mathcal{E}_{w_{\alpha,\beta,\gamma}}^n &= \delta_x e_{w_{\alpha,\beta,\gamma}}^n - \delta_x E_{w_{\alpha,\beta,\gamma}}^n, & \delta_x \mathcal{H}_{w_{\alpha,\beta,\gamma}}^n &= \delta_x h_{w_{\alpha,\beta,\gamma}}^n - \delta_x H_{w_{\alpha,\beta,\gamma}}^n, \\ \delta_x \mathcal{E}_{w_{\alpha,\beta,\gamma}}^{n+\frac{1}{2}} &= \delta_x e_{w_{\alpha,\beta,\gamma}}^{n*} - \delta_x E_{w_{\alpha,\beta,\gamma}}^n, & \delta_x \mathcal{H}_{w_{\alpha,\beta,\gamma}}^{n+\frac{1}{2}} &= \delta_x h_{w_{\alpha,\beta,\gamma}}^{n*} - \delta_x H_{w_{\alpha,\beta,\gamma}}^n,\end{aligned}$$

where  $\mathcal{E}^n = (\mathcal{E}_x^n, \mathcal{E}_y^n, \mathcal{E}_z^n)$  and  $\mathcal{H}^n = (\mathcal{H}_x^n, \mathcal{H}_y^n, \mathcal{H}_z^n)$ . Then subtracting the  $\delta_x$ -ADI-FDTD scheme (i.e. the ADI-FDTD scheme applied by  $\delta_x$ ) from the discrete form of the Maxwell's equations (see (4.2)-(4.13)) leads to the following system of error equations:

**Error- $\delta_x$ -Stage 1:**

$$\begin{aligned}\delta_x \mathcal{E}_x^{n+\frac{1}{2}} - \frac{\Delta t}{2\varepsilon} \delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}} &= \delta_x \mathcal{E}_x^n - \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_y^n + \frac{\Delta t}{2} \delta_x \beta_x^{n+\frac{1}{2}} \Big|_{i,j,k}, \\ \delta_x \mathcal{E}_y^{n+\frac{1}{2}} - \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_x^{n+\frac{1}{2}} &= \delta_x \mathcal{E}_y^n - \frac{\Delta t}{2\varepsilon} \delta_x \delta_x \mathcal{H}_z^n + \frac{\Delta t}{2} \delta_x \beta_y^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}, \\ \delta_x \mathcal{E}_z^{n+\frac{1}{2}} - \frac{\Delta t}{2\varepsilon} \delta_x \delta_x \mathcal{H}_y^{n+\frac{1}{2}} &= \delta_x \mathcal{E}_z^n - \frac{\Delta t}{2\varepsilon} \delta_x \delta_y \mathcal{H}_x^n + \frac{\Delta t}{2} \delta_x \beta_z^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}, \\ \delta_x \mathcal{H}_x^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu} \delta_x \delta_z \mathcal{E}_y^{n+\frac{1}{2}} &= \delta_x \mathcal{H}_x^n - \frac{\Delta t}{2\mu} \delta_x \delta_y \mathcal{E}_z^n + \frac{\Delta t}{2} \delta_x \xi_x^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}, \\ \delta_x \mathcal{H}_y^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu} \delta_x \delta_x \mathcal{E}_z^{n+\frac{1}{2}} &= \delta_x \mathcal{H}_y^n - \frac{\Delta t}{2\mu} \delta_x \delta_z \mathcal{E}_x^n + \frac{\Delta t}{2} \delta_x \xi_y^{n+\frac{1}{2}} \Big|_{i,j,k+\frac{1}{2}}, \\ \delta_x \mathcal{H}_z^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu} \delta_x \delta_y \mathcal{E}_x^{n+\frac{1}{2}} &= \delta_x \mathcal{H}_z^n - \frac{\Delta t}{2\mu} \delta_x \delta_x \mathcal{E}_y^n + \frac{\Delta t}{2} \delta_x \xi_z^{n+\frac{1}{2}} \Big|_{i,j+\frac{1}{2},k}.\end{aligned}$$

Error- $\delta_x$ -Stage 2 can be obtained similarly. For example, the first error equation is given by

$$\delta_x \mathcal{E}_x^{n+1} + \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_y^{n+1} = \delta_x \mathcal{E}_x^{n+\frac{1}{2}} + \frac{\Delta t}{2\varepsilon} \delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta_x \beta_x^{n+\frac{1}{2}} \Big|_{i,j,k}.$$

We also need the following system of error equations for the ADI-FDTD scheme, which is obtained by repeating the argument above and letting  $\delta_x = I$  (the identity

operator): **Error-Stage 1**:

$$\begin{aligned}
\varepsilon \mathcal{E}_x^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_y \mathcal{H}_z^{n+\frac{1}{2}} &= \varepsilon \mathcal{E}_x^n - \frac{\Delta t}{2} \delta_z \mathcal{H}_y^n + \frac{\varepsilon \Delta t}{2} \beta_x^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k}, \\
\varepsilon \mathcal{E}_y^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_z \mathcal{H}_x^{n+\frac{1}{2}} &= \varepsilon \mathcal{E}_y^n - \frac{\Delta t}{2} \delta_x \mathcal{H}_z^n + \frac{\varepsilon \Delta t}{2} \beta_y^{n+\frac{1}{2}} \Big|_{i,j+\frac{1}{2},k}, \\
\varepsilon \mathcal{E}_z^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_x \mathcal{H}_y^{n+\frac{1}{2}} &= \varepsilon \mathcal{E}_z^n - \frac{\Delta t}{2} \delta_y \mathcal{H}_x^n + \frac{\varepsilon \Delta t}{2} \beta_z^{n+\frac{1}{2}} \Big|_{i,j,k+\frac{1}{2}}, \\
\mu \mathcal{H}_x^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_z \mathcal{E}_y^{n+\frac{1}{2}} &= \mu \mathcal{H}_x^n - \frac{\Delta t}{2} \delta_y \mathcal{E}_z^n + \frac{\mu \Delta t}{2} \xi_x^{n+\frac{1}{2}} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}, \\
\mu \mathcal{H}_y^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_x \mathcal{E}_z^{n+\frac{1}{2}} &= \mu \mathcal{H}_y^n - \frac{\Delta t}{2} \delta_z \mathcal{E}_x^n + \frac{\mu \Delta t}{2} \xi_y^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k+\frac{1}{2}}, \\
\mu \mathcal{H}_z^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta_y \mathcal{E}_x^{n+\frac{1}{2}} &= \mu \mathcal{H}_z^n - \frac{\Delta t}{2} \delta_x \mathcal{E}_y^n + \frac{\mu \Delta t}{2} \xi_z^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}.
\end{aligned}$$

**Error-Stage 2** can be obtained similarly. For example, the first equation in this stage is given by

$$\mathcal{E}_x^{n+1} + \frac{\Delta t}{2\varepsilon} \delta_z \mathcal{H}_y^{n+1} = \mathcal{E}_x^{n+\frac{1}{2}} + \frac{\Delta t}{2\varepsilon} \delta_y \mathcal{H}_z^{n+\frac{1}{2}} + \frac{\Delta t}{2} \beta_x^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k}.$$

In the above error equations,  $\mathcal{E}_{w\alpha,\beta,\gamma}^m$ ,  $\mathcal{H}_{w\alpha,\beta,\gamma}^m$ ,  $\beta_w^{\bar{n}}$  and  $\xi_w^{\bar{n}}$  with  $w = x, y, z, m = n, \bar{n}$ , can be regarded as those in **Error- $\delta_x$ -Stages 1 and 2** with  $\delta_x$  replaced with the identity operator  $I$ . For example,

$$\begin{aligned}
\mathcal{E}_x^{n+\frac{1}{2}} &= e_x^{n*} - E_x^n \Big|_{i+\frac{1}{2},j,k}, \quad \mathcal{H}_z^{n+\frac{1}{2}} = h_z^{n*} - H_z^n \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}, \\
e_x^{n*} &= \frac{1}{2}(e_x^{n+1} + e_x^n) + \frac{\Delta t}{4\varepsilon} \delta_z (h_y^{n+1} - h_y^n) \Big|_{i+\frac{1}{2},j,k}, \\
h_z^{n*} &= \frac{1}{2}(h_z^{n+1} + h_z^n) + \frac{\Delta t}{4\mu} \delta_x (e_y^{n+1} - e_y^n) \Big|_{i+\frac{1}{2},j+\frac{1}{2},k}.
\end{aligned}$$

$e_y^{n*}$ ,  $e_z^{n*}$ ,  $h_x^{n*}$  and  $h_y^{n*}$  can be similarly defined. Note that  $\beta_w^{n+\frac{1}{2}}$  and  $\xi_w^{n+\frac{1}{2}}$  with  $w = x, y, z$  are the truncation error terms for each equation of the equivalent form of the ADI-FDTD scheme (i.e. the equivalent form of the  $\delta_x$ -ADI-FDTD scheme above with  $\delta_x = I$ ). For example,

$$\begin{aligned}
\beta_x^{n+\frac{1}{2}} &= \delta_t e_x^{n+\frac{1}{2}} - \frac{1}{\varepsilon} \bar{\delta}_t \left[ \delta_y h_z^{n+\frac{1}{2}} - \delta_z h_y^{n+\frac{1}{2}} \right] - \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_y \delta_x \delta_t e_y^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k}, \\
\xi_x^{n+\frac{1}{2}} &= \delta_t h_x^{n+\frac{1}{2}} - \frac{1}{\mu} \bar{\delta}_t \left[ \delta_z e_y^{n+\frac{1}{2}} - \delta_y e_z^{n+\frac{1}{2}} \right] - \frac{(\Delta t)^2}{4\mu\varepsilon} \delta_z \delta_x \delta_t h_z^{n+\frac{1}{2}} \Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}.
\end{aligned}$$

If the solution of the Maxwell equations is suitably smooth then the following estimates hold

$$(4.16) \quad |\beta_{x_{i,j,k}}^{n+\frac{1}{2}}| + |\beta_{y_{i,j,k}}^{n+\frac{1}{2}}| + |\beta_{z_{i,j,k}}^{n+\frac{1}{2}}| \leq C_{eh} [(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2],$$

$$(4.17) \quad |\xi_{x_{i,j,k}}^{n+\frac{1}{2}}| + |\xi_{y_{i,j,k}}^{n+\frac{1}{2}}| + |\xi_{z_{i,j,k}}^{n+\frac{1}{2}}| \leq C_{eh} [(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2],$$

where  $C_{eh}$  is a positive constant depending on  $\varepsilon, \mu$  and the upper bounds of the derivatives of  $\mathbf{e}, \mathbf{h}$ .

we now estimate (4.1) by making use of these error equations. Multiplying both sides of the first and second three equations in **Error- $\delta_x$ -Stage 1** by  $\sqrt{\varepsilon}$  and  $\sqrt{\mu}$ , respectively, and taking the square of the resulting equations, we will obtain six equations which are similar to the six equations (3.10)-(3.15) in the proof of Theorem 3.2, but each equation has one extra term  $f_i^n$  ( $i = 1, \dots, 6$ ) which related to the local truncation error term. For example, the first equation is given by

$$\begin{aligned} & \varepsilon(\delta_x \mathcal{E}_x^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}})^2 - \Delta t \delta_x \mathcal{E}_x^{n+\frac{1}{2}} \cdot \delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}} \\ &= \varepsilon(\delta_x \mathcal{E}_x^n)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x \delta_z \mathcal{H}_y^n)^2 - \Delta t \delta_x \mathcal{E}_x^n \cdot \delta_x \delta_z \mathcal{H}_y^n + f_1^n \Big|_{i,j,k}, \end{aligned}$$

where

$$f_1^n = \frac{\varepsilon}{4} (\Delta t)^2 (\delta_x \beta_x^{n+\frac{1}{2}})^2 + \varepsilon \Delta t (\delta_x \mathcal{E}_x^n - \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_y^n) \delta_x \beta_x^{n+\frac{1}{2}}.$$

The other terms  $f_i^n$  ( $i = 2, \dots, 6$ ) can be defined similarly and obviously.

Similarly, we can obtain six equations from **Error- $\delta_x$ -Stage 2**. For example, the first equation in this stage can be obtained from (4.16):

$$\begin{aligned} & \varepsilon(\delta_x \mathcal{E}_x^{n+1})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x \delta_z \mathcal{H}_y^{n+1})^2 + \Delta t \delta_x \mathcal{E}_x^{n+1} \cdot \delta_x \delta_z \mathcal{H}_y^{n+1} + \bar{f}_1^{n+1} \\ &= \varepsilon(\delta_x \mathcal{E}_x^{n+\frac{1}{2}})^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu(\delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}})^2 + \Delta t \delta_x \mathcal{E}_x^{n+\frac{1}{2}} \cdot \delta_x \delta_y \mathcal{H}_z^{n+\frac{1}{2}} \Big|_{i+\frac{1}{2},j,k}, \end{aligned}$$

where

$$\bar{f}_1^{n+1} = \frac{\varepsilon}{4} (\Delta t)^2 (\delta_x \beta_x^{n+\frac{1}{2}})^2 - \varepsilon \Delta t (\delta_x \mathcal{E}_x^{n+1} + \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_y^{n+1}) \delta_x \beta_x^{n+\frac{1}{2}}.$$

The other terms  $\bar{f}_i^{n+1}$  ( $i = 2, \dots, 6$ ) corresponding to the other five equations are similar to  $\bar{f}_1^{n+1}$ .

Using exactly the same argument as in the proof of Theorem 3.2 we arrive at

$$\begin{aligned} (4.18) \quad & \|\delta_x \mathcal{E}^{n+1}\|_E^2 + \|\delta_x \mathcal{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^{n+1}\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^{n+1}\|_E^2) \\ & + \|\mathcal{E}^{n+1}\|_I^2 + \|\mathcal{H}^{n+1}\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_2^h \mathcal{H}^{n+1}\|_I^2 + \|\delta_1^h \mathcal{E}^{n+1}\|_I^2) \\ &= \|\delta_x \mathcal{E}^n\|_E^2 + \|\delta_x \mathcal{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^n\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^n\|_E^2) \\ & + \|\mathcal{E}^n\|_I^2 + \|\mathcal{H}^n\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_2^h \mathcal{H}^n\|_I^2 + \|\delta_1^h \mathcal{E}^n\|_I^2) \\ & + F^{n,n+1} \Delta v + G^{n,n+1} \Delta y \Delta z, \end{aligned}$$

where  $\Delta v = \Delta x \Delta y \Delta z$  and for  $n \geq 0$  we have

$$\begin{aligned} F^{n,n+1} &= \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} [f_1^n - \bar{f}_1^{n+1}]_{i,j,k} + \sum_{i=1}^{I-2} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} [f_2^n - \bar{f}_2^{n+1}]_{i+\frac{1}{2},j+\frac{1}{2},k} \\ &+ \sum_{i=1}^{I-2} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} [f_3^n - \bar{f}_3^{n+1}]_{i+\frac{1}{2},j,k+\frac{1}{2}} + \sum_{i=1}^{I-2} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} [f_4^n - \bar{f}_4^{n+1}]_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} [f_5^n - \bar{f}_5^{n+1}]_{i,j,k+\frac{1}{2}} + \sum_{i=1}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} [f_6^n - \bar{f}_6^{n+1}]_{i,j+\frac{1}{2},k} \\
& := I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n,
\end{aligned}$$

$$\begin{aligned}
G^{n,n+1} &= (G_1^{n,n+1} + G_2^{n,n+1} + G_3^{n,n+1})/\Delta x, \\
G_1^{n,n+1} &= \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ \varepsilon \Delta t (\mathcal{E}_z^{n+1} + \mathcal{E}_z^n) + \frac{(\Delta t)^2}{2} \delta_y (\mathcal{H}_x^{n+1} - \mathcal{H}_x^n) \right] \beta_z^{n+\frac{1}{2}} \Big|_{i', j, \bar{k}}, \\
G_2^{n,n+1} &= \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left[ \varepsilon \Delta t (\mathcal{E}_y^{n+1} + \mathcal{E}_y^n) + \frac{(\Delta t)^2}{2} \delta_x (\mathcal{H}_z^{n+1} - \mathcal{H}_z^n) \right] \beta_y^{n+\frac{1}{2}} \Big|_{i', \bar{j}, k}, \\
G_3^{n,n+1} &= \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left[ \mu \Delta t (\mathcal{H}_x^{n+1} + \mathcal{H}_x^n) + \frac{(\Delta t)^2}{2} \delta_y (\mathcal{E}_z^{n+1} + \mathcal{E}_z^n) \right] \xi_x^{n+\frac{1}{2}} \Big|_{i', \bar{j}, \bar{k}}.
\end{aligned}$$

We now estimate each term in  $F^{n,n+1}$  and  $G^{n,n+1}$ . Noting that by the initial condition  $\mathcal{H}_y^0 = \mathcal{E}_x^0 = 0$ , we have

$$\begin{aligned}
\sum_{l=0}^n I_1^l &= \sum_{l=0}^n \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \varepsilon \Delta t \left[ \delta_x (\mathcal{E}_x^{l+1} + \mathcal{E}_x^l) + \frac{\Delta t}{2\varepsilon} \delta_x \delta_z (\mathcal{H}_y^{l+1} - \mathcal{H}_y^l) \right] \delta_x \beta_x^{l+\frac{1}{2}} \Big|_{i,j,k} \\
&\leq \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \left[ \frac{1}{2} \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_x \delta_z \mathcal{H}_y^{n+1})^2 + \frac{\varepsilon}{2} (\delta_x \mathcal{E}_x^{n+1})^2 \right. \\
&\quad \left. + \Delta t \sum_{l=0}^n \left\{ \varepsilon (\delta_x \mathcal{E}_x^l)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_x \delta_z \mathcal{H}_y^l)^2 \right\} \right]_{i,j,k} \\
(4.19) \quad &+ C\{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}.
\end{aligned}$$

Similar estimates can be obtained for  $I_i^n$  with  $i = 2, \dots, 6$ . Add these estimates together and use the definition of the discrete norms we get

$$\begin{aligned}
\sum_{l=0}^n F^{l,l+1} \Delta v &\leq \frac{1}{2} \frac{(\Delta t)^2}{4\mu\varepsilon} \left\{ \|\delta_x \delta_1^h \mathcal{E}^{n+1}\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^{n+1}\|_E^2 \right\} \\
&\quad + \frac{1}{2} (\|\delta_x \mathcal{E}^{n+1}\|_E^2 + \|\delta_x \mathcal{H}^{n+1}\|_H^2) \\
&\quad + \Delta t \sum_{l=0}^n \left( \|\delta_x \mathcal{E}^l\|_E^2 + \|\delta_x \mathcal{H}^l\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^l\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^l\|_E^2) \right) \\
(4.20) \quad &+ C\{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}.
\end{aligned}$$

Arguing similarly as in deriving (4.19) we can derive that

$$\begin{aligned}
(4.21) \quad \sum_{l=0}^n G_1^{l,l+1} &\leq \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ \frac{1}{2} \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_y \mathcal{H}_x^{n+1})^2 + \varepsilon (\mathcal{E}_z^{n+1})^2 \right. \\
&\quad \left. + \Delta t \sum_{l=0}^n \left( \varepsilon (\mathcal{E}_z^l)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\delta_y \mathcal{H}_x^l)^2 \right) \right]_{i', j, k+\frac{1}{2}} + C(\Delta)^4,
\end{aligned}$$



$$(4.22) \quad \sum_{l=0}^n G_2^{l,l+1} \leq \sum_{i'=1, I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{K-1} \left[ \frac{1}{2} \left( \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_x \mathcal{H}_z^{n+1})^2 + \varepsilon (\mathcal{E}_y^{n+1})^2 \right) \right. \\ \left. + \Delta t \sum_{l=0}^n \left( (\mathcal{E}_y^l)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_x \mathcal{H}_z^l)^2 \right) \right]_{i', j+\frac{1}{2}, k} + C(\Delta)^4;$$

$$(4.23) \quad \sum_{l=0}^n G_3^{l,l+1} \leq \sum_{i'=1, I-1} \sum_{j=1}^{J-1} \sum_{k=0}^{K-1} \left[ \frac{1}{2} \left( \frac{(\Delta t)^2}{4\mu\varepsilon} \varepsilon (\delta_y \mathcal{E}_z^{n+1})^2 + \mu (\mathcal{H}_x^{n+1})^2 \right) \right. \\ \left. + \Delta t \sum_{l=0}^n \left( \mu (\mathcal{H}_x^l)^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \varepsilon (\delta_y \mathcal{E}_z^l)^2 \right) \right]_{i', j+\frac{1}{2}, k+\frac{1}{2}} + C(\Delta)^4,$$

where  $(\Delta)^4 = (\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4$ . Combining (4.21)-(4.23) and using the definition of the discrete norms we obtain that

$$(4.24) \quad \sum_{n=0}^n G^{n,n+1} \Delta y \Delta z \leq \frac{1}{2} \left\{ \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_1^h \mathcal{E}^{n+1}\|_I^2 + \|\delta_2^h \mathcal{H}^{n+1}\|_I^2) + \|\mathcal{E}^{n+1}\|_I^2 + \|\mathcal{H}^{n+1}\|_I^2 \right\} \\ + \Delta t \sum_{l=0}^n \left( \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_1^h \mathcal{E}^l\|_I^2 + \|\delta_2^h \mathcal{H}^l\|_I^2) + \|\mathcal{E}^l\|_I^2 + \|\mathcal{H}^l\|_I^2 \right) + C(\Delta)^4.$$

By summing up both sides of (4.18) over the time levels  $n$  and using (4.20) and (4.24) it follows that

$$(4.25) \quad \|\delta_x \mathcal{E}^{n+1}\|_E^2 + \|\delta_x \mathcal{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^{n+1}\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^{n+1}\|_E^2) \\ + \|\mathcal{E}^{n+1}\|_I^2 + \|\mathcal{H}^{n+1}\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} [\|\delta_2^h \mathcal{H}^{n+1}\|_I^2 + \|\delta_1^h \mathcal{E}^{n+1}\|_I^2] \\ \leq C \Delta t \sum_{l=0}^n [\|\delta_x \mathcal{E}^l\|_E^2 + \|\delta_x \mathcal{H}^l\|_H^2 + \|\mathcal{E}^l\|_I^2 + \|\mathcal{H}^l\|_I^2 \\ + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^l\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^l\|_E^2 + \|\delta_2^h \mathcal{H}^l\|_I^2 + \|\delta_1^h \mathcal{E}^l\|_I^2)] + C(\Delta)^4.$$

The discrete Gronwall's Lemma 4.1 implies the estimate (4.1) with  $w = x$ . This completes the proof.  $\square$

If we apply the difference operators  $\delta_t$  and  $\delta_w \delta_t$  with  $w = x, y, z$  to the equations in the ADI-FDTD scheme and argue similarly as in the proof of Theorem 4.2, then we are able to obtain the following result.

**THEOREM 4.3.** *Suppose the exact solution  $\mathbf{e}, \mathbf{h}$  satisfies the same conditions as in Theorem 4.2. Then for any  $n \geq 0$  and any fixed  $T > 0$  there is a constant  $C$  independent of  $\Delta t, \Delta x, \Delta y, \Delta z$  such that for any  $w = x, y, z$ ,*

$$(4.26) \quad \|\delta_w \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_w \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_1^h \delta_w \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_2^h \delta_w \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_E^2) \\ + \|\delta_t \mathcal{E}^{n+\frac{1}{2}}\|_{L(w)}^2 + \|\delta_t \mathcal{H}^{n+\frac{1}{2}}\|_{L(w)}^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} [\|\delta_2^h \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_{L(w)}^2 + \|\delta_1^h \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_{L(w)}^2] \\ \leq C \{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\},$$

where  $L(x) = I$ ,  $L(y) = J$  and  $L(z) = K$ .

*Proof.* We only prove (4.26) for the case with  $w = x$ . The other cases with  $w = y, z$  can be proved similarly.

Repeating the argument in the proof of Theorem 4.2 with  $n$  being replaced by  $n - 1/2$  and each function  $U$  being replaced by  $\delta_t U$ , we are able to obtain that for any  $n \geq 1$ ,

$$\begin{aligned}
& \|\delta_x \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_x \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_x \delta_2^h \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_E^2) \\
& + \|\delta_t \mathcal{E}^{n+\frac{1}{2}}\|_I^2 + \|\delta_t \mathcal{H}^{n+\frac{1}{2}}\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \{ \|\delta_2^h \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_I^2 + \|\delta_1^h \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_I^2 \} \\
& \leq C \left\{ \|\delta_x \delta_t \mathcal{E}^{\frac{1}{2}}\|_E^2 + \|\delta_x \delta_t \mathcal{H}^{\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \delta_t \mathcal{E}^{\frac{1}{2}}\|_H^2 + \|\delta_x \delta_2^h \delta_t \mathcal{H}^{\frac{1}{2}}\|_E^2) \right. \\
& \quad + \|\delta_t \mathcal{E}^{\frac{1}{2}}\|_I^2 + \|\delta_t \mathcal{H}^{\frac{1}{2}}\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \{ \|\delta_2^h \delta_t \mathcal{H}^{\frac{1}{2}}\|_I^2 + \|\delta_1^h \delta_t \mathcal{E}^{\frac{1}{2}}\|_I^2 \} \\
& \quad \left. + (\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4 \right\}.
\end{aligned}$$

Since  $\mathcal{E}^0 = \mathcal{H}^0 = 0$ ,  $\delta_t \mathcal{E}^{1/2} = (1/\Delta t) \mathcal{E}^1$  and  $\delta_t \mathcal{H}^{1/2} = (1/\Delta t) \mathcal{H}^1$ , it is enough to prove

$$\begin{aligned}
(4.27) \quad & \|\delta_x \mathcal{E}^1\|_E^2 + \|\delta_x \mathcal{H}^1\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^1\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^1\|_E^2) \\
& + \|\mathcal{E}^1\|_I^2 + \|\mathcal{H}^1\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_2^h \mathcal{H}^1\|_I^2 + \|\delta_1^h \mathcal{E}^1\|_I^2) \\
& \leq C(\Delta t)^2 [(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4].
\end{aligned}$$

Letting  $n = 0$  in (4.18) and using the fact that  $\mathcal{E}^0 = \mathcal{H}^0 = 0$  we find that

$$\begin{aligned}
(4.28) \quad & \|\delta_x \mathcal{E}^1\|_E^2 + \|\delta_x \mathcal{H}^1\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_x \delta_1^h \mathcal{E}^1\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^1\|_E^2) \\
& + \|\mathcal{E}^1\|_I^2 + \|\mathcal{H}^1\|_I^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_2^h \mathcal{H}^1\|_I^2 + \|\delta_1^h \mathcal{E}^1\|_I^2) \\
& \leq F^{0,1} \Delta v + G^{0,1} \Delta y \Delta z,
\end{aligned}$$

where  $\Delta v = \Delta x \Delta y \Delta z$ . We now estimate each term of  $F^{0,1}$  and  $G^{0,1}$  slightly differently from the proof of Theorem 4.2 (cf. (4.19)-(4.24)) since we do not need to take the sum with respect to  $n$  here. For example, taking  $n = 0$  in (4.19) we have, on noting that  $\mathcal{E}_x^0 = 0$ , that

$$\begin{aligned}
I_1^0 &= \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \left[ \varepsilon \Delta t (\delta_x \mathcal{E}_x^1 + \frac{\Delta t}{2\varepsilon} \delta_x \delta_z \mathcal{H}_y^1) \delta_x \beta_x^{\frac{1}{2}} \right]_{i,j,k} \\
&\leq \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \left[ \frac{1}{2} \frac{(\Delta t)^2}{4\mu\varepsilon} \mu (\delta_x \delta_z \mathcal{H}_y^1)^2 + \frac{\varepsilon}{2} (\delta_x \mathcal{E}_x^1)^2 \right]_{i,j,k} \\
&\quad + C(\Delta t)^2 [(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4].
\end{aligned}$$

The other five terms  $I_i^0$ ,  $i = 2, \dots, 6$ , of  $F^{0,1}$  and  $G_i^{0,1}$ ,  $i = 1, 2, 3$ , can be estimated similarly. Then by the definition of the discrete norms it is derived (cf. (4.24)) that

$$F^{0,1} \Delta v + G^{0,1} \Delta y \Delta z \leq \frac{1}{2} \frac{(\Delta t)^2}{4\mu\varepsilon} \left\{ \|\delta_x \delta_1^h \mathcal{E}^1\|_H^2 + \|\delta_x \delta_2^h \mathcal{H}^1\|_E^2 \right\}$$

$$\begin{aligned}
& + \|\delta_1^h \mathcal{E}^1\|_I^2 + \|\delta_2^h \mathcal{H}^1\|_I^2 \} + \frac{1}{2} (\|\delta_x \mathcal{E}^1\|_E^2 + \|\delta_x \mathcal{H}^1\|_H^2 + \|\mathcal{E}^1\|_I^2 + \|\mathcal{H}^1\|_I^2) \\
& + C(\Delta t)^2 [(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4].
\end{aligned}$$

This together with (4.28) implies the estimate (4.27). The proof is thus complete.  $\square$

REMARK 4.4. It is a superconvergence result that the ADI-FDTD scheme is second-order convergent in space under the discrete  $H^1$  semi-norm (see Theorems 4.2 and 4.3). This is consistent with the superconvergence result of the semi-discrete Yee scheme established in [18] since both the ADI-FDTD scheme and the Yee scheme are based on the same spatial discretization technique.

Setting  $\delta_w = I$  with  $w = x, y, z$  in Theorems 4.2 and 4.3 gives the following optimal error estimates in the discrete  $L^2$  norm, which can be proved by arguing similarly as in the proof of Theorems 4.2 and 4.3 with  $\delta_w = I$ .

THEOREM 4.5. *Suppose the exact solution  $\mathbf{e}, \mathbf{h}$  satisfies the same conditions as in Theorem 4.2. Then for any  $n \geq 0$  and any fixed  $T > 0$  there is a constant  $C$  independent of  $\Delta t, \Delta x, \Delta y, \Delta z$  such that*

$$\begin{aligned}
(4.29) \quad & \|\mathcal{E}^{n+1}\|_E^2 + \|\mathcal{H}^{n+1}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_1^h \mathcal{E}^{n+1}\|_H^2 + \|\delta_2^h \mathcal{H}^{n+1}\|_E^2) \\
& \leq C\{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}, \\
(4.30) \quad & \|\delta_t \mathcal{E}^{n+\frac{1}{2}}\|_E^2 + \|\delta_t \mathcal{H}^{n+\frac{1}{2}}\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} (\|\delta_1^h \delta_t \mathcal{E}^{n+\frac{1}{2}}\|_H^2 + \|\delta_2^h \delta_t \mathcal{H}^{n+\frac{1}{2}}\|_E^2) \\
& \leq C\{(\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}.
\end{aligned}$$

**5. Divergence preserving property of the ADI-FDTD scheme.** As a consequence of the Maxwell equations (2.2)-(2.4), if the electric field  $\mathbf{e}$  and the magnetic field  $\mathbf{h}$  (multiplied with  $\varepsilon$  and  $\mu$  respectively) start out divergence free, they will remain so during wave propagation at any time and any place in the domain  $\Omega$ . In this section, we will prove that this property is preserved with the second-order accuracy in both space and time by the ADI-FDTD scheme.

THEOREM 5.1. *Let  $\mathbf{E}^n$  and  $\mathbf{H}^n$  be the solution of the ADI-FDTD scheme and let  $\nabla^h \cdot (\varepsilon \mathbf{E}^n)$  and  $\nabla^h \cdot (\mu \mathbf{H}^n)$  denote their discrete divergence:*

$$\begin{aligned}
\nabla^h \cdot (\varepsilon \mathbf{E}^n) &= \varepsilon (\delta_x E_x^n + \delta_y E_y^n + \delta_z E_z^n) \big|_{i,j,k}, \\
\nabla^h \cdot (\mu \mathbf{H}^n) &= \mu (\delta_x H_x^n + \delta_y H_y^n + \delta_z H_z^n) \big|_{\bar{i},\bar{j},\bar{k}}.
\end{aligned}$$

Then

$$(5.1) \quad \|\nabla^h \cdot (\varepsilon \mathbf{E}^n)\|^2 \leq C\{\|\nabla^h \cdot (\varepsilon \mathbf{e}^0)\|^2 + (\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\},$$

$$(5.2) \quad \|\nabla^h \cdot (\mu \mathbf{H}^n)\|^2 \leq C\{\|\nabla^h \cdot (\mu \mathbf{h}^0)\|^2 + (\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\},$$

where for a grid function  $U$ ,  $\|U\|^2 = \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \varepsilon(U_{i,j,k})^2 \Delta v$ .

*Proof.* We only prove (5.1). The inequality (5.2) can be proved similarly. From the property that  $\nabla \cdot (\varepsilon \mathbf{e}) = \nabla \cdot (\varepsilon \mathbf{e}^0)$  it follows that

$$\|\nabla^h \cdot (\varepsilon \mathbf{e}^n)\|^2 \leq \|\nabla^h \cdot (\varepsilon \mathbf{e}^0)\|^2 + C\{(\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}.$$

Thus, and by Theorem 4.2 it is derived that

$$\begin{aligned}\|\nabla^h \cdot (\varepsilon \mathbf{E}^n)\|^2 &= \|\nabla^h \cdot [\varepsilon(\mathbf{e}^n - \mathbf{E}^n + \mathbf{e}^n)]\|^2 \\ &\leq 2(\|\nabla^h \cdot (\varepsilon \mathcal{E}^n)\|^2 + \|\nabla^h \cdot (\varepsilon \mathbf{e}^n)\|^2) \\ &\leq C\{\|\nabla^h \cdot (\varepsilon \mathbf{e}^0)\|^2 + (\Delta t)^4 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\}\end{aligned}$$

for some generic constant  $C > 0$ , which is (5.1). The proof is thus complete.  $\square$

**6. Numerical experiments.** In this section we carry out numerical experiments to verify the energy identities, optimal error estimates and divergence preserving properties of the ADI-FDTD scheme.

**6.1. An exact solution and its energy conservation.** Consider the Maxwell equations (2.2)-(2.4) with  $\varepsilon = \mu = 1$ . The exact solution is given by

$$\begin{aligned}e_x &= \frac{1}{4}\sqrt{3}\cos(\sqrt{3}\pi t)\cos[\pi(1-x)]\sin[\pi(1-y)]\sin[\pi(1-z)], \\ e_y &= \frac{1}{2}\sqrt{3}\cos(\sqrt{3}\pi t)\sin[\pi(1-x)]\cos[\pi(1-y)]\sin[\pi(1-z)], \\ e_z &= -\frac{3}{4}\sqrt{3}\cos(\sqrt{3}\pi t)\sin[\pi(1-x)]\sin[\pi(1-y)]\cos[\pi(1-z)], \\ h_x &= -\frac{5}{4}\sin(\sqrt{3}\pi t)\sin[\pi(1-x)]\cos[\pi(1-y)]\cos[\pi(1-z)], \\ h_y &= \sin(\sqrt{3}\pi t)\cos[\pi(1-x)]\sin[\pi(1-y)]\cos[\pi(1-z)], \\ h_z &= \frac{1}{4}\sin(\sqrt{3}\pi t)\cos[\pi(1-x)]\cos[\pi(1-y)]\sin[\pi(1-z)].\end{aligned}$$

It is easy to verify that the above solution has the following energy conservation and divergence-free properties:

$$\begin{aligned}\|\mathbf{e}\|^2 &= \frac{21}{64}\cos^2(\sqrt{3}\pi t), \quad \|\mathbf{h}\|^2 = \frac{21}{64}\sin^2(\sqrt{3}\pi t), \quad \nabla \cdot \mathbf{e} = \nabla \cdot \mathbf{h} = 0, \\ E_{eh}^2 &= \|\mathbf{e}\|^2 + \|\mathbf{h}\|^2 = \frac{21}{64}, \quad E_{eht}^2 = \left\|\frac{\partial \mathbf{e}}{\partial t}\right\|^2 + \left\|\frac{\partial \mathbf{h}}{\partial t}\right\|^2 = \frac{63}{64}\pi^2, \\ E_{ehx}^2 &= \left\|\frac{\partial \mathbf{e}}{\partial x}\right\|^2 + \left\|\frac{\partial \mathbf{h}}{\partial x}\right\|^2 = \frac{21}{64}\pi^2, \quad E_{ehxt}^2 = \left\|\frac{\partial^2 \mathbf{e}}{\partial x \partial t}\right\|^2 + \left\|\frac{\partial^2 \mathbf{h}}{\partial x \partial t}\right\|^2 = \frac{63}{64}\pi^4,\end{aligned}$$

where for a vector function  $\mathbf{u} = (u_x, u_y, u_z)$  the  $L^2$ -norm  $\|\cdot\|$  is defined by

$$\|\mathbf{u}\|^2 = \int_{\Omega} P(\mathbf{u}) \left( |u_x|^2 + |u_y|^2 + |u_z|^2 \right) dx, \quad P(\mathbf{e}) = \varepsilon, \quad P(\mathbf{e}) = \mu.$$

$E_{ehw}$  and  $E_{ehwt}$  can be defined similarly. Note that  $E_{ehw} = E_{ehx}$ ,  $E_{ehwt} = E_{ehxt}$ ,  $w = y, z$ .

**6.2. Energy identities of the ADI-FDTD scheme.** Denote by  $\|\cdot\|_1$  and  $\|\cdot\|_2$  the two new norms defined in Theorems 3.2 and 3.4 respectively. Then

$$\begin{aligned}\|(\mathbf{E}^n, \mathbf{H}^n)\|_1^2 &= \|\delta_x \mathbf{E}^n\|_E^2 + \|\delta_x \mathbf{H}^n\|_H^2 + \|\mathbf{E}^n\|_I^2 + \|\mathbf{H}^n\|_I^2 \\ &\quad + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_x \delta_1^h \mathbf{E}^n\|_H^2 + \|\delta_x \delta_2^h \mathbf{H}^n\|_E^2 + \|\delta_2^h \mathbf{H}^n\|_I^2 + \|\delta_1^h \mathbf{E}^n\|_I^2 \right) \\ \|(\mathbf{E}^n, \mathbf{H}^n)\|_2^2 &= \|\mathbf{E}^n\|_E^2 + \|\mathbf{H}^n\|_H^2 + \frac{(\Delta t)^2}{4\mu\varepsilon} \left( \|\delta_2^h \mathbf{H}^n\|_E^2 + \|\delta_1^h \mathbf{E}^n\|_H^2 \right).\end{aligned}$$

Similarly, we can define  $\|(\delta_t \mathbf{E}^{\bar{n}}, \delta_t \mathbf{H}^{\bar{n}})\|_1^2$  and  $\|(\delta_t \mathbf{E}^{\bar{n}}, \delta_t \mathbf{H}^{\bar{n}})\|_2^2$ . To verify the four energy identities established in Subsection 3.3, we introduce the following notations:

$$EH_j^{n0} = \frac{\|(\mathbf{E}^n, \mathbf{H}^n)\|_j - \|(\mathbf{E}^0, \mathbf{H}^0)\|_j}{\|(\mathbf{E}^0, \mathbf{H}^0)\|_j}, \quad EH_1^n = \frac{\|(\mathbf{E}^n, \mathbf{H}^n)\|_1}{E_{ehx}}, \quad EH_2^n = \frac{\|(\mathbf{E}^n, \mathbf{H}^n)\|_2}{E_{eh}},$$

$$EH_{tj}^{n0} = \frac{\|(\delta_t \mathbf{E}^{n+\frac{1}{2}}, \delta_t \mathbf{H}^{n+\frac{1}{2}})\|_j - \|(\delta_t \mathbf{E}^{\frac{1}{2}}, \delta_t \mathbf{H}^{\frac{1}{2}})\|_j}{\|(\delta_t \mathbf{E}^{\frac{1}{2}}, \delta_t \mathbf{H}^{\frac{1}{2}})\|_j}, \quad EH_{tj}^n = \frac{\|(\delta_t \mathbf{E}^{\bar{n}}, \delta_t \mathbf{H}^{\bar{n}})\|_j}{E_{eht}}, \quad j = 1, 2.$$

Here,  $EH_j^{n0}$ ,  $EH_{tj}^{n0}$ ,  $j = 1, 2$ , represent the relative error of the two new energy norms of the solution to the ADI-FDTD scheme at time level  $t^n$  and at the initial time  $t^0$  or  $t^1$ , and  $EH_j^n$ ,  $EH_{tj}^n$ ,  $j = 1, 2$ , stand for the ratio of the two new energy norms of the solution of the ADI-FDTD scheme to the corresponding energy norms of the exact solution.

The computational results are presented in Table 6.1 for different time levels  $n$  ( $n = T/\Delta t$ ), where the solution to the ADI-FDTD scheme is computed with the spatial step sizes  $\Delta x = \Delta y = \Delta z = 0.01$  and the time step size  $\Delta t = 0.01$  (note that the Courant number is  $\sqrt{3}$ ). From Table 6.1 it is seen that the four discrete energy identities shown in Theorems 3.2, 3.3, 3.4 and 3.5 are satisfied and the discrete energy of the approximate solution is almost equal to the corresponding ones of the exact solution (their ratio is almost equal to 1).

TABLE 6.1

*Relative error of the discrete energy of the ADI-FDTD solution at time level  $n$  and at the initial time, and the ratio of the discrete energy of the ADI-FDTD solution to the true energy of the exact solution under the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  at different time levels with  $\Delta x = \Delta y = \Delta z = \Delta t = 0.01$*

Energy\Time	n=100	n=400	n=800	n=1600	n=2000
$EH_1^{n0}$	3.157e-13	4.780e-14	2.783e-13	2.537e-13	1.646e-13
$EH_1^n$	1.00008	1.00008	1.00008	1.00008	1.00008
$EH_{t1}^{n0}$	1.627e-12	1.518e-12	1.428e-12	1.478e-12	1.559e-12
$EH_{t1}^n$	0.99979	0.99979	0.99979	0.99979	0.99979
$EH_2^{n0}$	1.397e-13	2.808e-13	3.432e-13	2.254e-13	2.167e-13
$EH_2^n$	1.0001	1.0001	1.0001	1.0001	1.0001
$EH_{t2}^{n0}$	1.457e-12	1.585e-12	1.744e-12	1.438e-12	1.634e-12
$EH_{t2}^n$	0.99984	0.99984	0.99984	0.99984	0.99984

To verify the energy identities without the perturbation terms, define

$$\|(\mathbf{E}^n, \mathbf{H}^n)\|_{1*}^2 := \|\delta_x \mathbf{E}^n\|_E^2 + \|\delta_x \mathbf{H}^n\|_H^2 + \|\mathbf{E}^n\|_I^2 + \|\mathbf{H}^n\|_I^2,$$

$$\|(\mathbf{E}^n, \mathbf{H}^n)\|_{2*}^2 := \|\mathbf{E}^n\|_E^2 + \|\mathbf{H}^n\|_H^2.$$

Similarly, let  $EH_{j*}^{n0}$ ,  $EH_{tj*}^{n0}$ ,  $j = 1, 2$ , represent the relative error of the two new energy norms without the perturbation terms of the solution to the ADI-FDTD scheme at time level  $t^n$  and at the initial time  $t^0$  or  $t^1$  and let  $EH_{j*}^n$ ,  $EH_{tj*}^n$ ,  $j = 1, 2$ , stand for the ratio of the two new energy norms without the perturbation terms of the solution of the ADI-FDTD scheme to the exact ones.

Table 6.2 shows similar results to those in Table 6.1 with the four discrete energy identities without the perturbation terms. These new discrete energy norms can be seen as the discrete forms of the physical energies. From these values we can see that

the energy identities without the perturbation terms are still satisfied and that the discrete energies of the ADI-FDTD solution are almost equal to the corresponding ones of the exact solution.

TABLE 6.2

*Relative error of the discrete energy of the ADI-FDTD solution at time level  $n$  and at the initial time, and the ratio of the discrete energy of the ADI-FDTD solution to the physical energy of the exact solution under the norms  $\|\cdot\|_{1*}$  and  $\|\cdot\|_{2*}$  at different time levels with  $\Delta x = \Delta y = \Delta z = \Delta t = 0.01$*

Energy\Time	n=100	n=400	n=800	n=1600	n=2000
$EH_{1*}^{n0}$	1.592e-14	3.140e-13	1.730e-14	1.413e-13	1.070e-14
$EH_{1*}^n$	0.99996	0.99996	0.99996	0.99996	0.99996
$EH_{t1*}^{n0}$	1.942e-13	3.359e-13	2.508e-13	2.899e-13	4.480e-13
$EH_{t1*}^n$	0.99967	0.99967	0.99967	0.99967	0.99967
$EH_{2*}^{n0}$	1.463e-13	1.421e-13	1.531e-13	2.105e-13	6.590e-15
$EH_{2*}^n$	1.0000	1.0000	1.0000	1.0000	1.0000
$EH_{t2*}^{n0}$	9.150e-14	9.050e-14	2.309e-13	2.269e-13	3.705e-15
$EH_{t2*}^n$	0.99971	0.99984	0.99971	0.99971	0.99971

**6.3. Error and convergence rate of the ADI-FDTD scheme.** To verify the convergence rate of the ADI-FDTD scheme under different norms, let

$$\begin{aligned}
\mathcal{EH}_1^n &= \|(\mathcal{E}^n, \mathcal{H}^n)\|_1 / E_{ehx}, & \mathcal{EH}_{t1}^n &= \|(\delta_t \mathcal{E}^{n+\frac{1}{2}}, \delta_t \mathcal{H}^{n+\frac{1}{2}})\|_1 / E_{ehxt}, \\
\mathcal{EH}_2^n &= \|(\mathcal{E}^n, \mathcal{H}^n)\|_2 / E_{eh}, & \mathcal{EH}_{t2}^n &= \|(\delta_t \mathcal{E}^{n+\frac{1}{2}}, \delta_t \mathcal{H}^{n+\frac{1}{2}})\|_2 / E_{eht}, \\
\mathcal{EH}_0^n &= (\|\mathcal{E}^n\|_E^2 + \|\mathcal{H}^n\|_H^2) / E_{eh}.
\end{aligned}$$

Table 6.3 gives the relative error in different norms of the ADI-FDTD solution at different time levels with the time step size  $\Delta t = 0.01$ , the spatial step sizes  $\Delta x = \Delta y = \Delta z = 0.01$  and the Courant number  $\sqrt{3}$ . The results demonstrate that even for a large Courant number and at a long time, e.g.  $T = 20$ , the ADI-FDTD solution is still stable and accurate in different norms.

TABLE 6.3

*Relative error in the norms  $\|\cdot\|_1$ , and  $\|\cdot\|_2$  of the ADI-FDTD solution at different time with  $\Delta x = \Delta y = \Delta z = 0.01$  and  $\Delta t = 0.01$*

Error\Time	n=100	n=400	n=800	n=1600	n=2000
$\mathcal{EH}_1^n$	9.091e-4	3.580e-3	7.161e-3	1.432e-2	1.790e-2
$\mathcal{EH}_{t1}^n$	9.162e-4	3.585e-3	7.159e-3	1.431e-2	1.789e-2
$\mathcal{EH}_2^n$	3.214e-4	1.266e-3	2.532e-3	5.063e-3	6.329e-3
$\mathcal{EH}_{t2}^n$	9.159e-4	3.585e-3	7.195e-3	1.431e-2	1.789e-2

The convergence rates in time and space of the ADI-FDTD scheme are given in Tables 6.4-6.9 under different norms. The results show that the convergence rate in the norms  $\|\cdot\|_1$ ,  $\|\cdot\|_2$  and  $\|\cdot\|_0$  is second order in space and time, which confirm the theoretical results. Note that the second-order convergence in time and space under the norm  $\|\cdot\|_1$  is a superconvergence result.

TABLE 6.4

Convergence rate in time under the norm  $\|\cdot\|_1$  at  $T = 1$  with  $\Delta x = \Delta y = \Delta z = 0.01$ 

$\Delta t$	$\mathcal{EH}_1^n$	Rate	$\mathcal{EH}_{t1}^n$	Rate
0.05	1.749e-2		1.706e-2	
0.04	1.127e-2	1.971	1.108e-2	1.934
0.025	2.978e-3	1.938	2.975e-3	1.913
0.02	2.978e-3	1.881	2.975e-3	1.863

TABLE 6.5

Convergence rate in space under the norm  $\|\cdot\|_1$  at  $T = 1$  with  $\Delta t = 0.001$ 

$\Delta x = \Delta y = \Delta z$	$\mathcal{EH}_1^N$	Rate	$\mathcal{EH}_{t1}^N$	Rate
0.025	1.405e-3		1.430e-3	
0.02	9.016e-4	1.987	9.221e-4	1.966
0.01	2.305e-4	1.968	3.141e-4	1.554

TABLE 6.6

Convergence rate in time under the norm  $\|\cdot\|_2$  at  $T = 1$  with  $\Delta x = \Delta y = \Delta z = 0.01$ 

$\Delta t$	$\mathcal{EH}_2^n$	Rate	$\mathcal{EH}_{t2}^n$	Rate
0.050	6.185e-3		1.706e-2	
0.040	3.984e-3	1.971	1.108e-2	1.934
0.025	2.978e-3	1.938	2.975e-3	1.913
0.020	1.053e-3	1.881	2.975e-3	1.863

TABLE 6.7

Convergence rate in space under the norm  $\|\cdot\|_2$  at  $T = 1$  with  $\Delta t = 0.001$ 

$\Delta x = \Delta y = \Delta z$	$\mathcal{EH}_2^N$	Rate	$\mathcal{EH}_{t2}^N$	Rate
0.025	4.968e-4		1.428e-3	
0.020	3.188e-4	1.988	9.165e-4	1.987
0.010	8.149e-5	1.968	2.342e-4	1.968

**6.4. Divergence of the numerical electric field.** To verify the discrete divergence-free property of the ADI-FDTD scheme, we introduce two norms for divergence:

$$\text{Div}_{L^\infty} = \max_{i,j,k} \{ \varepsilon |\delta_x E_x^M + \delta_y E_y^M + \delta_z E_z^M|_{i,j,k} \},$$

$$\text{Div}_{L^2} = \left( \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} \varepsilon |\delta_x E_x^M + \delta_y E_y^M + \delta_z E_z^M|_{i,j,k}^2 \Delta v \right)^{1/2}.$$

Table 6.10 gives the values of the above norms at different times  $T = 1, 4, 8, 16$  and  $20$ , where  $M\Delta t = T$ . The results show that the discrete divergence is almost zero, which means that the ADI-FDTD scheme is of the approximate divergence preserving property.

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TABLE 6.8  
Convergence rate in time under the norm  $\|\cdot\|_0$  at  $T = 1$  with  $\Delta x = \Delta y = \Delta z = 0.01$

$\Delta t$	$\ \mathcal{E}^n\ _E$	Rate	$\ \mathcal{H}^n\ _E$	Rate
0.050	1.914e-2		1.595e-2	
0.040	1.232e-2	1.975	1.031e-2	1.956
0.025	4.955e-3	1.938	4.157e-3	1.933
0.020	3.260e-3	1.876	2.730e-3	1.883

TABLE 6.9  
Convergence rate in space under the norm  $\|\cdot\|_0$  at  $T = 1$  with  $\Delta t = 0.001$

$\Delta x = \Delta y = \Delta z$	$\ \mathcal{E}^N\ _E$	Rate	$\ \mathcal{H}^N\ _E$	Rate
0.025	4.968e-4		1.7064e-2	
0.020	3.188e-4	1.988	1.1082e-2	1.934
0.010	8.149e-5	1.968	2.975e-3	1.913

TABLE 6.10  
Divergence error of the numerical electric field  $\mathbf{E}$  in  $L^\infty$  and  $L^2$  norms at different times

Error/Time	T=1	T=4	T=8	T=16	T=20
$\text{Div}_{L^\infty}$	7.060e-13	1.421e-12	1.943e-12	2.956e-12	2.912e-12
$\text{Div}_{L^2}$	7.421e-14	1.540e-13	2.176e-13	3.085e-13	3.451e-13

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